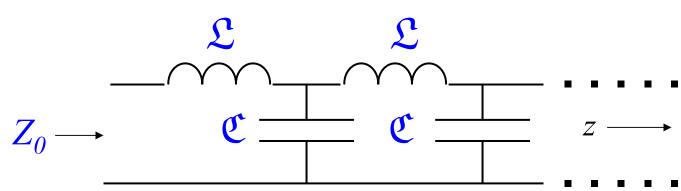
Circuit, or Lumped-Element, Model for T-Lines

Why?

- Enables T-line analysis using only circuit equations, not Maxwell's equations
  - T-line represented as circuit schematic
  - Lumped-Element model commonly used
- Can be used for any TEM T-line geometry
- Uses circuit parameters that we can measure or calculate
  - Capacitance & inductance per unit length
  - Conductor resistance and dielectric conductivity

Lumped-Element Model for Lossless Transmission Line



The change in voltage with z $\left(\frac{\partial V(z)}{\partial z}\right)$ is given by the

voltage drop across the inductors: $\frac{\partial V(z)}{\partial z} = -(j\omega\mathfrak{L})I(z)$

The change in current with z $\left(\frac{\partial I(z)}{\partial z}\right)$ is given by

the current in the capacitors: $\frac{\partial I(z)}{\partial z} = -(j\omega\mathfrak{C})V(z)$

Analysis of Circuit Equation

Take the derivative with respect to z of both sides of the equation for $V(z)$:

$$\frac{\partial^2 V(z)}{\partial z^2} = -(j\omega\mathfrak{L})\frac{\partial I(z)}{\partial z}$$

Now substitute for $\frac{\partial I(z)}{\partial z}$ using the equation on the previous slide

$$\frac{\partial^2 V(z)}{\partial z^2} = -(j\omega\mathfrak{L})\frac{\partial I(z)}{\partial z} = (j\omega\mathfrak{L})(j\omega\mathfrak{C})V(z) = -\omega^2\mathfrak{L}\mathfrak{C}V(z)$$

$$\Rightarrow \frac{\partial^2 V(z)}{\partial z^2} + \omega^2\mathfrak{L}\mathfrak{C}V(z) = 0$$

This is in the form of the wave equation, which we know how to solve:

$V(z) = V_0 e^{-j\gamma z}$ which satisfies our equation if $\gamma = \omega\sqrt{\mathfrak{L}\mathfrak{C}}$

So, $V(z) = V_0 e^{-j\omega\sqrt{\mathfrak{L}\mathfrak{C}}z}$ and $I(z) = I_0 e^{-j\omega\sqrt{\mathfrak{L}\mathfrak{C}}z}$

Solving for Z_0

Now, we can go back to our original circuit equations to find Z_0 (we can do this with either of our starting equations, but we'll use $V(z)$ here):

$$\frac{\partial V(z)}{\partial z} = -(j\omega\mathfrak{L})I(z) \Rightarrow \frac{\partial V_0 e^{-j\omega\sqrt{\mathfrak{L}\mathfrak{C}}z}}{\partial z} = -(j\omega\mathfrak{L})I_0 e^{-j\omega\sqrt{\mathfrak{L}\mathfrak{C}}z}$$

$$\frac{\partial V_0 e^{-j\omega\sqrt{\mathfrak{L}\mathfrak{C}}z}}{\partial z} = \cancel{-j\omega}\sqrt{\mathfrak{L}\mathfrak{C}}V_0 \cancel{e^{-j\omega\sqrt{\mathfrak{L}\mathfrak{C}}z}} = (\cancel{-j\omega}\mathfrak{L})I_0 \cancel{e^{-j\omega\sqrt{\mathfrak{L}\mathfrak{C}}z}}$$

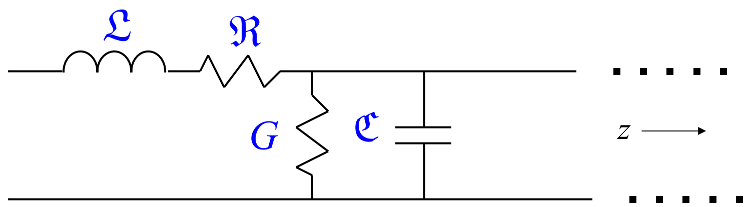
We can rearrange the above equation to get the ratio of V to I:

$$Z_0 = \frac{V_0}{I_0} = \frac{\mathfrak{L}}{\sqrt{\mathfrak{L}\mathfrak{C}}} = \sqrt{\frac{\mathfrak{L}}{\mathfrak{C}}}\ \Omega$$

Thus, we can calculate the characteristic impedances of the three transmission line structures discussed in class, since we know their inductances and capacitances per unit length.

Lumped-Element Model for Lossy Transmission Line

By introducing a series resistance term, R, and a shunt conductance term, G, we can account for loss in the conductor and dielectric.



$\mathfrak{R}$ is the resistance per unit length in the conductors and is a function of frequency

G is the conductance per unit length of the dielectric between the conductors and is typically a function of frequency

Analysis of Circuit Equation

The change in voltage with z $\left(\frac{\partial V(z)}{\partial z}\right)$ is given by the voltage drop across the inductors and resistors: $\frac{\partial V(z)}{\partial z} = -\left(j\omega\mathfrak{L} + \mathfrak{R}\right)I(z)$

The change in current with z $\left(\frac{\partial I(z)}{\partial z}\right)$ is given by the current in the capacitors and dielectric: $\frac{\partial I(z)}{\partial z} = -\left(j\omega\mathfrak{C} + \mathfrak{G}\right)V(z)$

Analysis of Lossy Circuit Equation

Following the same approach as we used for the lossless line, we can take the derivative with respect to z of both sides of the equation for $V(z)$ and then substitute for the derivative of $I(z)$:

$$\frac{\partial^2 V(z)}{\partial z^2} = -\left(j\omega\mathfrak{L} + \mathfrak{R}\right)\frac{\partial I(z)}{\partial z} = \left(j\omega\mathfrak{L} + \mathfrak{R}\right)\left(j\omega\mathfrak{C} + \mathfrak{G}\right)V(z)$$

$$\Rightarrow \quad \frac{\partial^2 V(z)}{\partial z^2} + \left(j\omega\mathfrak{L} + \mathfrak{R}\right)\left(j\omega\mathfrak{C} + \mathfrak{G}\right)V(z) = 0$$

This is in the form of the wave equation, which we know how to solve:

$V(z) = V_0 e^{-j\gamma z}$ which satisfies our equation

if $\gamma = \sqrt{\left(j\omega\mathfrak{L} + \mathfrak{R}\right)\left(j\omega\mathfrak{C} + \mathfrak{G}\right)}$

We can use this as we did before to get the ratio of V to I:

$$Z_0 = \frac{V_0}{I_0} = \frac{\mathfrak{L}}{\sqrt{\mathfrak{L}\mathfrak{C}}} = \sqrt{\frac{\mathfrak{R} + j\omega\mathfrak{L}}{\mathfrak{G} + jw\mathfrak{C}}} \ \Omega$$