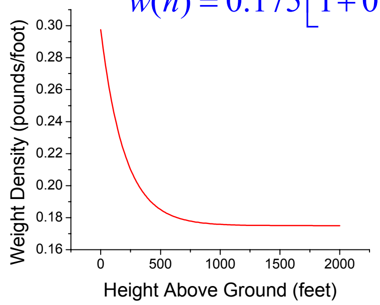
Integrating Functions Over a Line (or Contour)

Example: When dry, a climbing rope weighs 0.175 #/foot. The weight density for a wet rope hanging vertically is:

$$w(h) = 0.175\left[1 + 0.7e^{-0.005h}\right]\#\text{ per foot}$$



Find the weight of a wet 2000' rope hanging vertically

Note: a dry rope 2000' long would weigh 2000 feet x 0.175 #/foot = 350 #

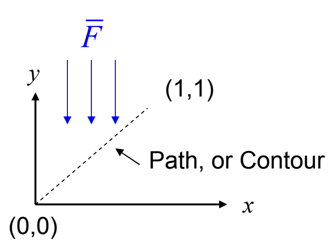
The weight of a wet rope is:

$$\text{Total weight} = \int_0^h w(h)dh = 0.175\int_0^{2000}\left(1 + 0.7e^{-.005h}\right)dh$$

$$= 0.175\left(h - 140e^{-.005h}\right)\Big|_0^{2000} = 0.175\left(2000 - 0.0064 + 140\right) = 374.5\#$$

Contour Integration with Vectors

Example: if the force acting on a particle is $\overline{F} = -\hat{a}_y$ Newtons, how much work is required to move the particle from (0,0) to (1,1) along the path $y = x$?



W = work = force times distance, where force is the force tangent to the path.

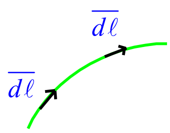
In this example, the angle between the path and the force is 45° over the entire path:

$F_{TAN} = \left|\overline{F}\right|\cos 45°$ The length of the path is $\sqrt{2} \Rightarrow$ work $= \sqrt{2}\cos 45° = 1$ Nm

To handle more general cases, where the vector or path varies, we will need to integrate. To do this, define a vector differential element, $\overline{d\ell}$

Differential Length Vector

$\overline{d\ell}$ is an infintesimally-short vector tangent to a curve or line



In general, for Cartesian coordinates

$$\overline{d\ell} = dx\hat{a}_x + dy\hat{a}_y + dz\hat{a}_z$$

This can be put in terms of *dx, dy,* or *dz* only to facilitate integration. The length of the differential length vector is:

$$\left|\overline{d\ell}\right| = d\ell = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$$

Vector Contour Integration

Needed to sum the tangential component of a vector function along a path. Generally of the form:

$$\int_C \overline{F}\cdot\overline{d\ell} \text{ or } \oint \overline{F}\cdot\overline{d\ell}$$

Where the second form is for a closed contour. Note that any closed contour necessarily defines a surface. The dot product provides a scalar representing the component of the vector in the direction of the contour.

Example

If $\overline{F} = -2\hat{a}_x + \hat{a}_y$ find the work required to go from (0,0) to (2,6) along the line $y = 3x$

Vector Contour Integral Example

$Work = \int_C F_T d\ell$ F_T = Tangential Force

$$dW = \overline{F} \cdot \overline{d\ell} = \left|\overline{F}\right| \cos\theta \left|\overline{d\ell}\right|$$

$$\overline{F} \cdot \overline{d\ell} = \left(-2\hat{a}_x + \hat{a}_y\right) \cdot \left(dx\hat{a}_x + dy\hat{a}_y + dz\hat{a}_z\right) = -2dx + dy$$

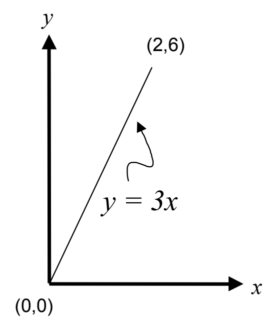
since y = 3x on the contour, $\frac{dy}{dx} = 3 \quad \Rightarrow \quad dy = 3dx$

$$\overline{F} \cdot \overline{d\ell} = -2dx + 3dx = dx$$

(could have put in terms of dy also)

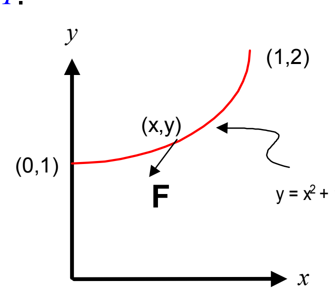
$$Work = \int_{(0,0)}^{(2,6)} \overline{F} \cdot \overline{dl} = \int_0^2 dx = 2$$

so :



Vector Contour Integral Example #2

A particle is attracted towards the origin with a force equal to kr, where r is the distance to the origin. How much work is required to move that particle from (0,1) to (1,2) along the path $y = x^2 + 1$.



Solution

First- find the force vector.

Since $x\hat{a}_x + y\hat{a}_y$ points from the origin to the point (x, y), then $-x\hat{a}_x - y\hat{a}_y$ points from (x, y) to the origin.

Vector Contour Integral Example #2 (2)

The length of this vector is $\sqrt{x^2 + y^2} = r$, so

$\overline{F} = k(-x\hat{a}_x - y\hat{a}_y)$

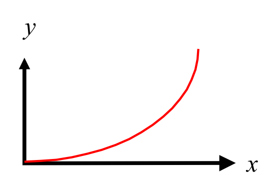
Work $= \int_{(0,1)}^{(1,2)} \overline{F} \cdot \overline{d\ell}$ and $\overline{F} \cdot \overline{d\ell} = k(-x\hat{a}_x - y\hat{a}_y) \cdot (dx\hat{a}_x + dy\hat{a}_y)$

$= k(-xdx - ydy) \Rightarrow$ Work $= \int_{(0,1)}^{(1,2)} k(-xdx - ydy)$

$$= k\left[-\int_0^1 xdx - \int_1^2 ydy\right] = -k\left(0.5 + 1.5\right) = -2k$$

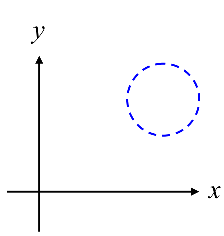
Open and Closed Contours

If a contour does not enclose a surface, it is an <u>open</u> contour:



Example: $y = x^2$

If a contour does enclose a surface, it is a <u>closed</u> contour:



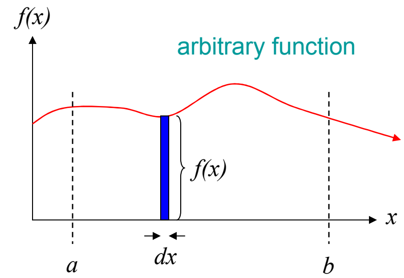
Example: $(x-3)^2+(y-2)^2 = 1$

Tutorial on Double Integrals

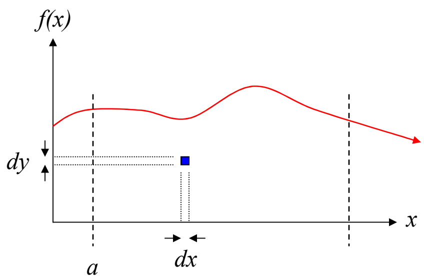
First consider a single integral

Range of x

$$\int_a^b \underbrace{f(x)dx}_{\text{area of differential element}} = \text{Total Area}$$



We can also express this as a double integral



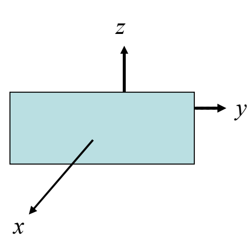
$$\int_a^b \int_0^{f(x)} dy\,dx = \int_a^b y\Big|_0^{f(x)} dx$$

$$\int_a^b f(x)dx \quad \text{Same as before}$$

Vector Surface Integration

<u>Example:</u> For the fluid velocity given below, find the volume of fluid per second passing through the portion of the $x=1$ plane that extends from -1 to 1 on z and -2 to 2 on y.

$\overline{V} = 2x\hat{a}_x + y^3\hat{a}_y + x^2 z\hat{a}_z$ m/sec



Only the x-component of velocity will pass through the surface so we only need V_x.

$V_x = 2x = 2$ m/sec on the $x = 1$ plane

Volume/sec $= \iint_S V_x(m/s)ds(m^2)$

Since V_x is constant on the surface we do not need to integrate

$$\iint_S V_x ds = V_x \times area = 2(m/s)\times 8(m^2) = 16m^3/\sec$$

Differential Surface Vector

Needed when integrating a vector quantity passing through a surface

$\overline{ds}$ is a vector that is normal to the surface. If the surface is closed, $\overline{ds}$ points out of the surface.

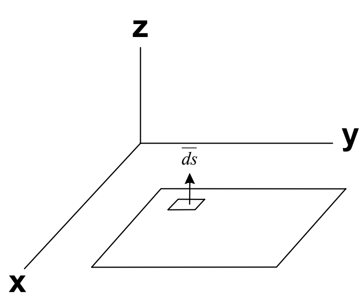
In general, the field passing through a surface is given by:

$$\iint_s \overline{V} \cdot \overline{ds} \text{ or } \oiint \overline{V} \cdot \overline{ds} \text{ if the surface is closed}$$

Note: a closed surface defines a volume

Surface Vector Integration Example #1

Find the rate of fluid flow through the surface on the z=0 plane defined by the region x = 1 to 3 and y = 2 to 4 if fluid flow is defined by: $\overline{V} = y^3\hat{a}_x + y^4(z+29)^2\hat{a}_y - x^2\hat{a}_z$ m/sec



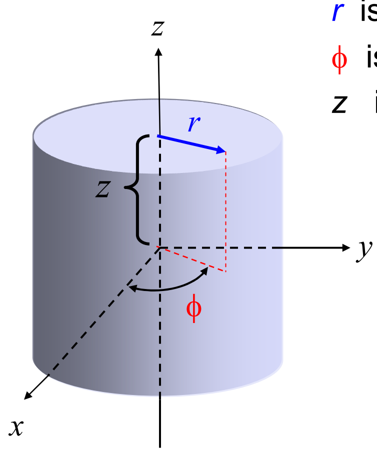
In this example, only the z component of fluid velocity will pass through this surface.

$$\overline{ds} = \pm ds\hat{a}_z = \pm dxdy\hat{a}_z$$

so $\overline{V} \cdot \overline{ds} = \pm x^2 dxdy$

$$\text{Flow rate } = \iint_s \overline{V} \cdot \overline{ds} = \int_2^4 \int_1^3 x^2 dxdy = 17\tfrac{1}{3} \quad m^3/\text{sec}$$

Cylindrical Coordinate System



r is the distance from the z-axis

ϕ is the angle with respect to the x-axis

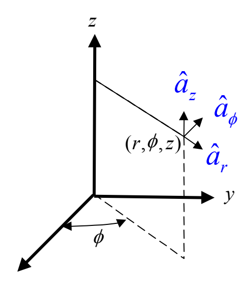
z is z

Any point in 3-D space can be defined by (r, ϕ, z)

Conversion to Cartesian Coordinates

$$x = r\cos\phi \qquad y = r\sin\phi \qquad z = z$$

Cylindrical Coordinate System

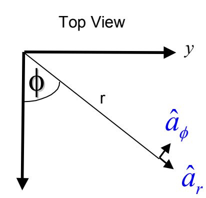


$\hat{a}_r$ is in the direction of increasing r

$\hat{a}_\phi$ is in the direction of increasing ϕ

$\hat{a}_z$ is in the direction of increasing z

$\hat{a}_r \perp \hat{a}_\phi \perp \hat{a}_z$



Differential surface element vectors

Curved surface $\overline{ds} = \pm\hat{a}_r r\,d\phi\,dz$

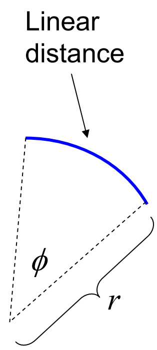
Top & bottom $\overline{ds} = \pm\hat{a}_z r\,d\phi\,dr$

Radial cut $\overline{ds} = \pm\hat{a}_\phi\,dr\,dz$

Differential volume element

$$dv = r\,d\phi\,dr\,dz$$

Comment on Radians



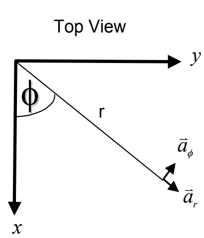
The linear distance traced out by ϕ is $r\phi$ if ϕ is in radians.

$$\phi \equiv \frac{\text{distance traveled at a constant radius}}{\text{radius}} \quad \text{(radians)}$$

If a wheel rolls one revolution, it covers a linear distance of $2\pi r$

Thus, there are 2π radians per revolution

Coordinate Conversion: Cylindrical-Cartesian



Using trigonometry:

$$x = r\cos\phi \Rightarrow \phi = \cos^{-1}\left(\frac{x}{r}\right)$$

$$y = r\sin\phi \Rightarrow \phi = \sin^{-1}\left(\frac{y}{r}\right)$$

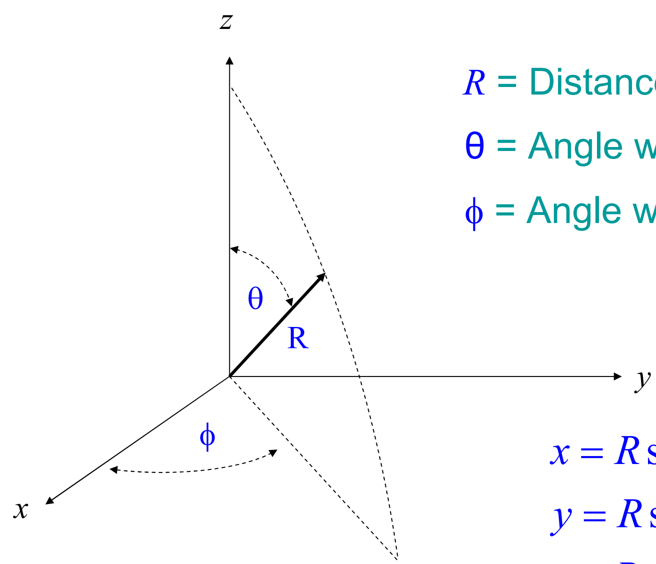
$$z = z$$

$$r = \sqrt{x^2 + y^2}$$

Example

If a point is defined by $(r, \phi, z) = (5, 30°, 8)$, that same point would be represented in Cartesian by $(x, y, z) = (4.33, 2.5, 8)$

Spherical Coordinates (R, θ, ϕ)



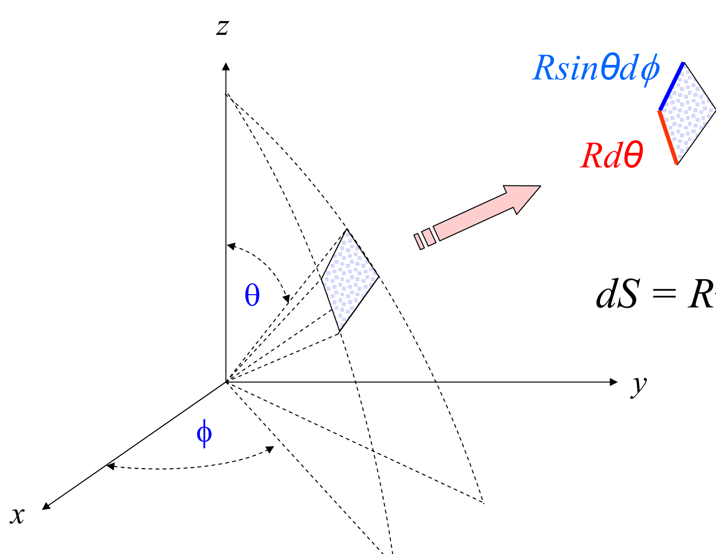
R = Distance from origin

θ = Angle with z-axis

ϕ = Angle with x-axis

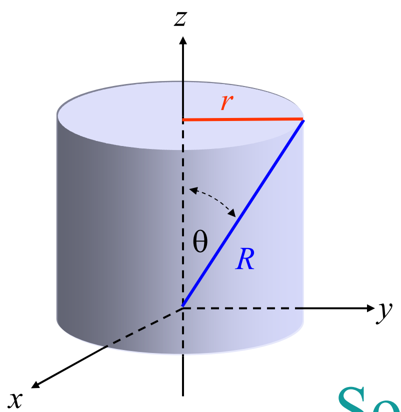
$$x = R\sin\theta\cos\phi$$
$$y = R\sin\theta\sin\phi$$
$$z = R\cos\theta$$

Spherical Differential Surface Element (dS)



$$dS = R^2 \sin\theta d\theta d\phi$$

Another Way To Look At *Rsinθ*

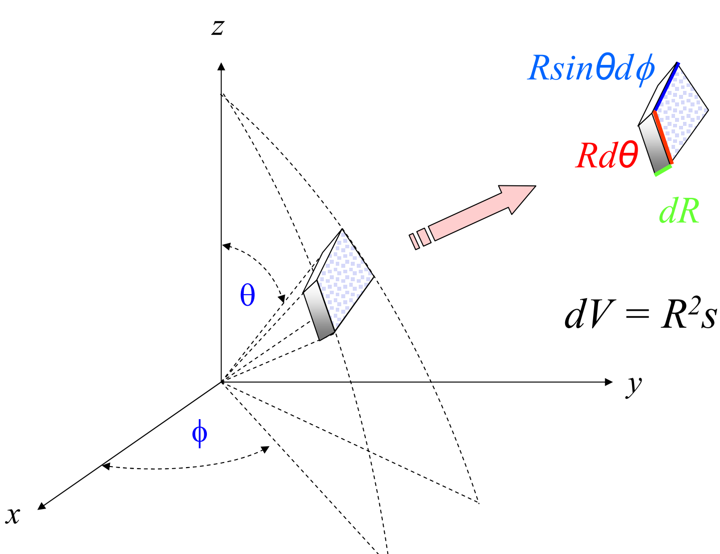


$$r = R\sin\theta$$

Cylindrical — Spherical

$$\text{So } r\,d\phi = R\sin\theta\,d\phi$$

Spherical Differential Volume Element (*dV*)



$$dV = R^2 \sin\theta\, d\theta\, d\phi\, dR$$

Spherical Examples

What is the surface area of a sphere?

$$\text{Area} = \iint ds = \int_0^{2\pi}\int_0^{\pi} R^2 \sin\theta \, d\theta \, d\phi = \int_0^{2\pi} -R^2 \cos\theta \Big|_0^{\pi} d\phi = 4\pi R^2$$

What is the volume of a sphere?

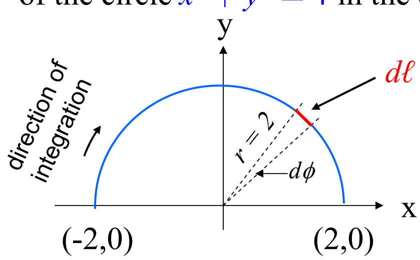
$$\text{Volume} = \iiint dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} R^2 \sin\theta \, dR \, d\theta \, d\phi = \int_0^{2\pi}\int_0^{\pi} \frac{R^3}{3}\Bigg|_0^{R} \sin\theta \, d\theta \, d\phi$$

$$= \frac{R^3}{3}\int_0^{2\pi} -\cos\theta \Big|_0^{\pi} d\phi = \frac{2R^3}{3}\phi \Bigg|_{\phi=0}^{2\pi} = \frac{4\pi R^3}{3}$$

Another Contour Integration Example

In a previous example, we integrated a function over a straight line.
In this example, we will integrate a function over a curved line.

Calculate $\oint f(x,y)\,d\ell$ over the contour defined by the upper half of the circle $x^2 + y^2 = 4$ in the direction shown, where $f(x,y) = x^2 y$



$d\ell = r d\phi = 2 d\phi$

$d\ell$ is a differential length element along the contour. So the first step is to define $d\ell$

Because the geometry of the contour is circular, we will want to use polar coordinates $\Rightarrow$ $d\ell = r d\phi = 2 d\phi$ on the contour, since $r = 2$ on the contour

Example (Continued)

Since we are defining our variable of integration, $d\ell$ in polar coordinates, we need to convert our function, $f(x,y)$, into polar coordinates (i.e., $f(r,\phi) = f(2,\phi)$ on the contour since $r = 2$ on the contour)

$$x = r\cos\phi = 2\cos\phi \qquad y = r\sin\phi = 2\sin\phi$$

$$f(x,y) = x^2 y \Rightarrow f(r,\phi) = f(2,\phi) = (2\cos\phi)^2(2\sin\phi)$$

So $\oint f(x,y)d\ell = \int_{\pi}^{0}(2\cos\phi)^2(2\sin\phi)2d\phi = \left.\frac{-16\cos^3\phi}{3}\right|_{\pi}^{0} = \frac{-32}{3}$

<u>Note</u> $\int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1}$