Maxwell's Equations

<u>Dynamic Fields:</u> for static fields, both the electric & magnetic fields are independent of each other. With time-varying (dynamic) fields, they are related by:

$$\nabla \times \overline{E} = -\frac{\partial \mu \overline{H}}{\partial t} \qquad \nabla \times \overline{H} = \overline{J} + \frac{\partial \varepsilon \overline{E}}{\partial t}$$

$$\nabla \cdot \varepsilon \overline{E} = \rho_v \qquad \nabla \cdot \overline{H} = 0$$

These are Maxwell's Equations in differential instantaneous form

Maxwell's Equations

$$\nabla \times \overline{E} = -j\omega\mu\overline{H} \qquad \nabla \times \overline{H} = \overline{J} + j\omega\varepsilon\overline{E}$$

$$\nabla \cdot \varepsilon \overline{E} = \rho_v \qquad \nabla \cdot \overline{H} = 0$$

These are Maxwell's Equations in differential phasor form

$$\oint \overline{E} \cdot \overline{d\ell} = -\frac{\partial}{\partial t} \iint \mu \overline{H} \cdot \overline{ds}$$

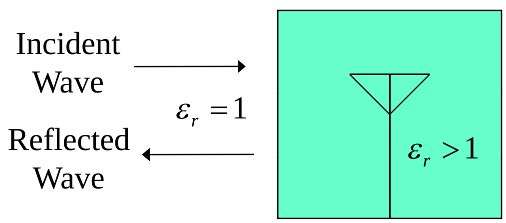
$$\oint \overline{H} \cdot \overline{d\ell} = \iint \overline{J} \cdot \overline{ds} + \frac{\partial}{\partial t} \iint \varepsilon \overline{E} \cdot \overline{ds}$$

$$\oiint \varepsilon \overline{E} \cdot \overline{ds} = \iiint \rho_v dv \qquad \oiint \overline{H} \cdot \overline{ds} = 0$$

These are Maxwell's Equations in integral form

Boundary Conditions

Enables us to determine the relationship between field values on either side of a material boundary. From this we can determine how signals will reflect from, or transmit into boundaries. This will be important in solving problems such as the one below where we wish to determine the effect on received signal strength caused by enclosing an antenna in a dielectric material.

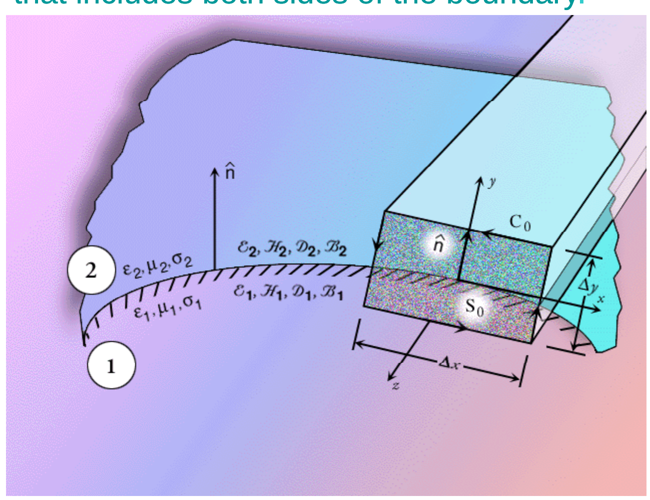


Not surprisingly, boundary conditions are found by applying Maxwell's equations

Boundary Conditions (finite conductivity)

If the media are finitely conducting, there will be no surface currents or surface charge densities. To analyze,consider a rectangular contour that includes both sides of the boundary.

To find the relationship between the electric fields in regions 1 & 2, apply Faraday's law and then take the limit as the contour shrinks to zero

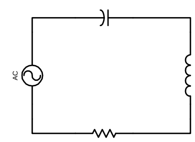


Boundary Conditions (2)

$$\oint_{C_0} \overline{E} \cdot \overline{d\ell} = -\frac{\partial}{\partial t} \iint_{S_0} \overline{B} \cdot \overline{ds}$$ Faraday's Law

By shrinking our contour, we will get to the point where the right side of the equality is negligible, which is equivalent to ignoring flux passing through the loop in Kirchoff's voltage law.

$$\oint_{C_0} \overline{E} \cdot \overline{d\ell} \Rightarrow \sum_i V_i = V_{source} + V_C + V_L + V_R$$

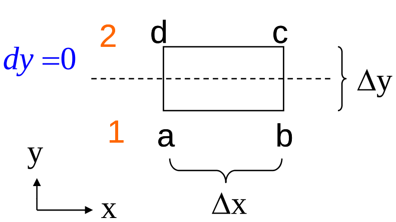


If the loop is very small with respect to a wavelength, the effects of magnetic flux passing through it will be negligible. For this case, assuming that the sum of the voltages around a loop is equal to zero is reasonable.

Boundary Conditions (3)

Break the closed contour into 4 integrals to represent the 4 segments of the rectangular contour.

$$\oint_{C_0} \overline{E} \cdot \overline{d\ell} = \int_a^b E_{T_1} dx + \int_b^c E_N dy + \int_c^d E_{T_2} + \int_d^a E_N dy = 0$$



Since our contour is very small, we can treat the integrands as being constant over Δx and Δy, which enables us to write:

$$\oint_{C_0} \overline{E} \cdot \overline{d\ell} = E_{T_1} \Delta x + E_N \Delta y - E_{T_2} \Delta x - E_N \Delta y = 0$$

The "T" subscript implies that the field component is tangent, or parallel, to the boundary. "N" denotes the normal component.

Boundary Conditions (4)

The integrals of the normal components over y will cancel since they will be of equal magnitude and opposite sign:

$$E_{T_1}\Delta x - E_{T_2}\Delta x = 0 \Rightarrow E_{T_1} = E_{T_2}$$

The tangential components of electric field are continuous across a boundary. Another way of stating this is:

> $\hat{n}\times\left(\bar{E}_2-\bar{E}_1\right)=0$ where $\hat{n}$ is the normal to the boundary and both σ_1 and σ_2 are finite.

To find the boundary conditions for the magnetic field, solve Ampere's law using the same contour.

$$\oint_{C_0}\bar{H}\cdot\overline{d\ell}=\frac{\partial}{\partial t}\iint_{S_0}\bar{D}\cdot\overline{ds}\qquad \text{Ampere's Law}$$

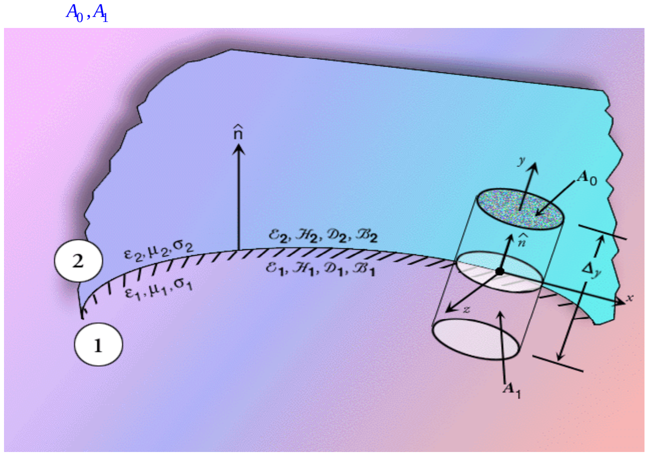
Since this is in the same mathematical form as Faraday's Law, we can conclude the following:

> $\hat{n}\times\left(\bar{H}_2-\bar{H}_1\right)=0$ where $\hat{n}$ is the normal to the boundary and both σ_1 and σ_2 are finite.

Boundary Conditions (5)

To find the boundary conditions for electric flux density, apply Gauss Law over the Gaussian surface shown.

$$\oiint_{A_0,A_1}\bar{D}\cdot\overline{ds}=\iiint\rho_v dv\qquad \text{Gauss Law}$$



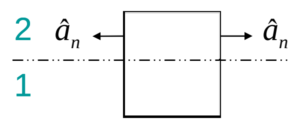
Boundary Conditions (6)

For a sufficiently small Gaussian surface $\iint_{\substack{curved\\surface}} \overline{D}\cdot\overline{ds} = 0$

because of symmetry and because $\boldsymbol{D}$ can be considered constant within each region.

However, we cannot assume the integral will sum to zero over the top and bottom of the surface, since they are in different media.

Side view of Gaussian surface



$$\oint\!\!\oint \overline{D}\cdot\overline{ds} = \iint_{top} \overline{D}\cdot\overline{ds} + \iint_{bottom} \overline{D}\cdot\overline{ds} = \iint_{top} D_{n_1} ds - \iint_{bottom} D_{n_2} ds$$

$= D_{n_1}\Delta s - D_{n_2}\Delta s$ where Δs is the surface area of the top & bottom.

The right side of Gauss equation, $\iiint \rho_v dv \to 0$ as the Gaussian surface shrinks. So, $D_{n_1}\Delta s - D_{n_2}\Delta s = 0 \Rightarrow D_{n_1} = D_{n_2}$

Boundary Conditions (7)

The normal components of electric flux density are continuous across a boundary. Another way of stating this is:

> $\hat{n}\cdot\left(\overline{D}_2 - \overline{D}_1\right) = 0$ where $\hat{n}$ is the normal to the boundary and both σ_1 and σ_2 are finite.

To find the boundary conditions for the magnetic flux density, solve Gauss Law for magnetic fields using the same Gaussian surface.

$$\oint\!\!\oint_{A_0, A_1} \overline{B}\cdot\overline{ds} = 0$$

Because this is mathematically identical to the electric flux density problem, the boundary conditions will also be identical:

> $\hat{n}\cdot\left(\overline{B}_2 - \overline{B}_1\right) = 0$ where $\hat{n}$ is the normal to the boundary and both σ_1 and σ_2 are finite.

Boundary Conditions (8)

Using the constitutive parameters, we can also write:

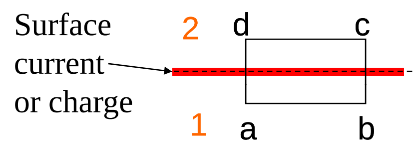
$\hat{n}\cdot\left(\varepsilon_2\bar{E}_2-\varepsilon_1\bar{E}_1\right)=0$ and $\hat{n}\cdot\left(\mu_2\bar{H}_2-\mu_1\bar{H}_1\right)=0$

where $\hat{n}$ is the normal to the boundary

and both σ_1 and σ_2 are finite.

If a source exists on a boundary, or if one of the media is perfectly conducting, we will need to account for the surface charges and/or currents on the boundary.

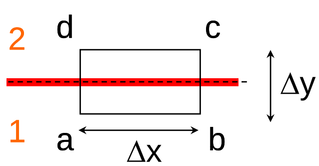
In this case, we will not be able to ignore the source(s), since the contour will enclose the source(s), no matter how small we make that contour.



Boundary Conditions (9)

To find the BC for magnetic field on either side of a perfectly-conducting boundary, again start with Ampere's Law:

$$\oint_{C_0}\bar{H}\cdot\overline{d\ell}=\frac{\partial}{\partial t}\iint_{S_0}\bar{D}\cdot\overline{ds}+\iint_{S_0}\bar{J}_s\cdot\overline{ds}$$ Ampere's Law



If $\bar{J}_s$ is a surface current , then the total conduction current passing through our contour $\iint_{S_0}\bar{J}_s\cdot\overline{ds}$ will be $J_{sz}\Delta x$ (assuming our contour is sufficiently small).

Also $\frac{\partial}{\partial t}\iint_{S_0}\bar{D}\cdot d\bar{S}\rightarrow 0$ as before

Solving the integral for an infinitesimally-small contour

$$\oint_{C_0}\bar{H}\cdot\overline{dl}=H_{T_1}\Delta x+\cancel{H_N\Delta y}-H_{T_2}\Delta x-\cancel{H_N\Delta y}=\bar{J}_s\cdot\hat{z}\Delta x=J_{sz}\Delta x$$

Boundary Conditions (10)

$H_{T_1} - H_{T_2} = J_{sz}$ for the geometry shown. The difference between the magnetic fields on either side of a boundary is equal to the surface current flowing on the boundary that are normal to H_T .

To put this statement into a more general (and usable) form, express our findings using vectors:

$(\bar{H}_1 - \bar{H}_2)\cdot\hat{x} = \bar{J}_s\cdot\hat{z} \Rightarrow (\bar{H}_1 - \bar{H}_2)\cdot\hat{x} - \bar{J}_s\cdot\hat{z} = 0$

Noting that: $\hat{x} = \hat{y}\times\hat{z}$ we can write the above equation as:

$(\bar{H}_1 - \bar{H}_2)\cdot(\hat{y}\times\hat{z}) - \bar{J}_s\cdot\hat{z} = 0$

A B C

Now, applying cyclic permutation to our scalar triple product: $\bar{A}\cdot\bar{B}\times\bar{C} = \bar{C}\cdot\bar{A}\times\bar{B}$

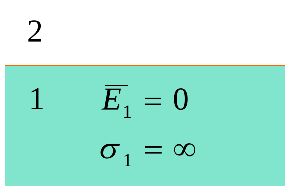
$\hat{z}\cdot[(\bar{H}_1 - \bar{H}_2)\times\hat{y}] - \bar{J}_s\cdot\hat{z} = 0 \quad \Rightarrow \quad \{[\hat{y}\times(\bar{H}_2 - \bar{H}_1)] - \bar{J}_s\}\cdot\hat{z} = 0$

since $\hat{y}$ is the normal to the boundary, we can write:

$\left[\hat{a}_n\times(\bar{H}_2 - \bar{H}_1)\right] = \bar{J}_s$

Boundary Conditions (11)

Now consider the BCs for electric field near a perfectly conducting boundary. Note that $|E|=0$ inside a perfect conductor and $\sigma=\infty$



$\hat{n}\times\left(\bar{E}_2 - \bar{E}_1\right) = \hat{n}\times\bar{E}_2 = 0 \Rightarrow E_{T_2} = 0$

There is no tangential component of electric field just above a perfect conductor

Electric field =0 ⇒ Magnetic field = 0

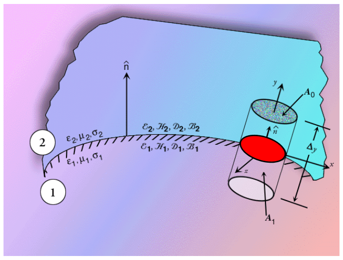
$$\nabla\times\bar{E}_1 = 0 = -\frac{\partial\bar{B}_1}{\partial t} \Rightarrow \bar{B}_1 = 0 \Rightarrow \bar{H}_1 = 0$$

When a perfect conductor is illuminated by an electromagnetic field, Lorentz forces cause charges to move on the surface, giving rise to a surface current density, defined by:

$$\hat{a}_n\times\bar{H}_2 = \bar{J}_s \Rightarrow H_{T_2} = J_s$$

Boundary Conditions (12)

To determine the BCs for the normal components of ***E*** and ***D*** we need to consider our Gaussian surface again. In this example, there exists a surface charges on the boundary



Again, we apply Gauss Law, but this time the right side does not vanish as the closed surface shrinks:

$$\oiint_{A_0, A_1} \bar{D} \cdot \overline{ds} = \iiint \rho_v dv \quad \text{Gauss Law}$$

Following a similar approach as before:

$$\oiint_{A_0, A_1} \bar{D} \cdot \overline{ds} = \iint_{A_0} \bar{D} \cdot \overline{ds} = \iint_{top} \bar{D} \cdot \overline{ds} + \iint_{bottom} \bar{D} \cdot \overline{ds} = \iint_{top} D_{n_1} ds - \iint_{bottom} D_{n_2} ds$$

$= D_{n2}\Delta s - D_{n1}\Delta s$ where Δs is the surface area of the top & bottom.

The right side of Gauss equation, $\iiint \rho_v dv = \rho_s \Delta s$

So, $D_{n2}\Delta s - D_{n1}\Delta s = \rho_v \Delta s \;\Rightarrow D_{n2} - D_{n1} = \rho_s$

Boundary Conditions (13)

In more general terms:

$\hat{n} \cdot \left(\bar{D}_2 - \bar{D}_1\right) = \rho_s \;\Rightarrow \hat{n} \cdot \left(\varepsilon_2 \bar{E}_2 - \varepsilon_1 \bar{E}_1\right) = \rho_s$
where $\hat{n}$ is the normal to the surface
and both σ_1 and σ_2 are finite.

If region 1 is perfectly conducting:

$$\hat{n} \cdot \bar{D}_2 = \rho_s \;\Rightarrow \hat{n} \cdot \varepsilon_2 \bar{E}_2 = \rho_s \Rightarrow E_{n2} = \frac{\rho_s}{\varepsilon_2}$$

Example: a perfectly-conducting surface is defined by $x^2 - 4y + 4z^2 = 4$ has a surface charge density $\rho_s = x + 3y - 2z$ Coul/m^2. Find the vector electric field at the point (2,1,1).

Solution: $\bar{E}$ will be $\perp$ to the surface, where $\hat{a}_n = \frac{\nabla \Phi}{|\nabla \Phi|} = \frac{4\hat{a}_x - 4\hat{a}_y + 8\hat{a}_z}{9.78}$

$\rho_s = 3$ at (2,1,1) $\Rightarrow \bar{E} = \frac{\rho_s \hat{a}_n}{\varepsilon} = \frac{12\hat{a}_x - 12\hat{a}_y + 24\hat{a}_z}{9.78\varepsilon}$

Boundary Condition Example

The boundary between two media exists on the x-y plane, with region 1 being defined by $z > 0$ and region 2 defined by $z < 0$. If the relative permittivity is 1 in region 1 and it is 4 in region 2, find the electric field in region 2 given the electric field in region 1 below: $\bar{E}_1 = 5\hat{a}_x - 2\hat{a}_y + 8\hat{a}_z$ Volts/m

Solution: Since the tangential components of $\boldsymbol{E}$ are the same on either side of the boundary, the x and y components of $\boldsymbol{E}$ in region 2 will be the same as in region 1.

The normal component of $\boldsymbol{D}$ (D_z) is the same on either side of a boundary, so $\varepsilon_1 E_{z1} = \varepsilon_2 E_{z2}$ (recalling that $\boldsymbol{D} = \varepsilon \boldsymbol{E}$)

Thus: $E_{z2} = \frac{\varepsilon_1}{\varepsilon_2} E_{z1} = \frac{1}{4} 8 = 2$

We can now write: $\bar{E}_2 = 5\hat{a}_x - 2\hat{a}_y + 2\hat{a}_z$ Volts/m

Boundary Condition Example #2

Region 1 is defined by $f(x,y,z) = -x + 5y - 6z > 4$, and region 2 is defined by $f(x,y,z) < 4$.

If $\bar{E}_1 = 3\hat{a}_x - 2\hat{a}_y + \hat{a}_z$ find $\bar{E}_2$ given that $\varepsilon_{r1} = 1$ and $\varepsilon_{r2} = 5$

The unit vector normal to the boundary is $\hat{a}_n = \pm \frac{\nabla f(x,y,z)}{|\nabla f(x,y,z)|} = \pm \left(\frac{-\hat{a}_x + 5\hat{a}_y - 6\hat{a}_z}{\sqrt{62}} \right)$

Pick either + or -, and reference both $\boldsymbol{E_1}$ and $\boldsymbol{E_2}$ to the same normal

The vector component of $\bar{E}_1$ normal to the boundary is:

$$\bar{E}_{N1} = \left(\bar{E}_1 \cdot \hat{a}_n \right) \hat{a}_n = \left(0.306\hat{a}_x - 1.53\hat{a}_y - 1.839\hat{a}_z \right)$$

Since the normal components of $\bar{D}$ are equal:

$$\bar{E}_{N2} = \varepsilon_1 / \varepsilon_2 \, \bar{E}_{N1} = \left(0.0612\hat{a}_x - .306\hat{a}_y - 0.3768\hat{a}_z \right)$$

The tangential components of $\bar{E}$ are equal and are given by:

$$\bar{E}_{T1} = \bar{E}_{T2} = \left(\bar{E} - \bar{E}_{N1} \right) = \left(2.69\hat{a}_x - 0.47\hat{a}_y - 0.839\hat{a}_z \right)$$

Thus: $\bar{E}_2 = \left(\bar{E}_{T2} + \bar{E}_{N2} \right) = \left(2.751\hat{a}_x - 0.776\hat{a}_y - 1.215\hat{a}_z \right)$