

## Dissipative (Lossy) Media

In a lossy medium ( $\sigma > 0$ ), a wave will lose energy by heating up that medium, and the fields will decrease exponentially in the direction of propagation. Returning to the wave equation, and including the conductivity term, we get:

$$\nabla^2 \bar{E} + (j\omega\mu\sigma - \omega^2\mu\varepsilon) \bar{E} = \nabla^2 \bar{E} + \gamma^2 \bar{E} = 0$$

$E_0 e^{\pm \gamma z}$  will continue to be a solution, and  $\gamma$  will be a complex number  $\Rightarrow$  phase shift and attenuation

$$\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t} = \bar{J} + j\omega \bar{D} = \bar{J} + j\omega \varepsilon \bar{E} = \sigma \bar{E} + j\omega \varepsilon \bar{E} = j\omega \left( \varepsilon - j \frac{\sigma}{\omega} \right) \bar{E}$$

$$\gamma^2 = -\omega^2 \mu \left( \varepsilon - j \frac{\sigma}{\omega} \right) \Rightarrow \gamma = \alpha + j\beta \quad \eta = \sqrt{\frac{\mu}{\left( \varepsilon - j \frac{\sigma}{\omega} \right)}} = |\eta| e^{j\phi}$$

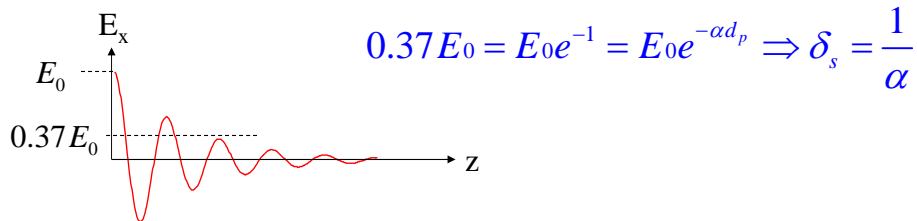
## Complex Wavenumber & Penetration Depth

The imaginary part of  $\gamma$  accounts for phase shift as the wave propagates, and the real part accounts for attenuation.

Consider an  $x$ -polarized wave propagating in the  $z$  direction:

$$\begin{aligned} E_x(z) &= E_0 e^{-\gamma z} = E_0 e^{-(\alpha + j\beta)z} = \underbrace{E_0 e^{-\alpha z}}_{\text{Amplitude}} \underbrace{e^{-j\beta z}}_{\text{Phase}} \\ &\Rightarrow E_0 \cos(\omega t - \beta z) e^{-\alpha z} \end{aligned}$$

Skin Depth ( $\delta_s$ ): the distance at which  $E$  is reduced to 37% of its incident value. This is also called penetration depth.



## Exact & Approximate Formulas

Exact

$$\gamma = j\omega \sqrt{\mu \left( \epsilon - j \frac{\sigma}{\omega} \right)} = j\omega \sqrt{\mu \epsilon} \left( 1 - j \frac{\sigma}{\epsilon \omega} \right)^{1/2} \quad \alpha = \frac{\omega \sqrt{\mu \epsilon}}{\sqrt{2}} \left[ \sqrt{1 + \left( \frac{\sigma}{\omega \epsilon} \right)^2} - 1 \right]^{1/2}$$

Good Conductor

$$\alpha = \beta \approx \sqrt{\pi f \mu \sigma} \quad \eta \approx \sqrt{\frac{j\omega \mu}{\sigma}}$$

Good Dielectric (low-loss dielectric)

$$\alpha \approx \frac{\omega}{2} \sqrt{\mu \epsilon} \left( \frac{\sigma}{\omega \epsilon} \right) \quad \beta \approx \omega \sqrt{\mu \epsilon} \left[ 1 + \frac{1}{8} \left( \frac{\sigma}{\omega \epsilon} \right)^2 \right]$$

$$\eta = \sqrt{\frac{\mu}{\epsilon}} \left\{ \left[ 1 - \frac{3}{8} \left( \frac{\sigma}{\omega \epsilon} \right)^2 \right] + j \frac{1}{2} \left( \frac{\sigma}{\omega \epsilon} \right) \right\}$$

## Loss Tangent

Loss Tangent is often used to determine how lossy a medium is at a particular frequency. It is the ratio of the conduction current to the displacement current.

$$\text{Loss Tangent} \equiv \frac{\sigma E}{\epsilon \omega E}$$

Example: what is the skin depth and loss tangent in bottom round steak at microwave oven frequencies (2.45 GHz)?  $\sigma = 1 \text{ S/m}$  &  $\epsilon_r = 40$

$$\alpha = \frac{\omega \sqrt{\mu \epsilon}}{\sqrt{2}} \left[ \sqrt{1 + \left( \frac{\sigma}{\omega \epsilon} \right)^2} - 1 \right]^{1/2} = 28.8 \text{ Nepers/m so } \delta_s = \frac{1}{\alpha} = \frac{1}{28.8} \text{ m} = 0.034 \text{ m} = 1.3''$$

$$\text{loss tangent} = \frac{\sigma}{\omega \epsilon} = \frac{1}{(2\pi \times 2.45 \times 10^9) (40 \times 8.85 \times 10^{-12})} = 0.184$$

## Loss Tangent Examples

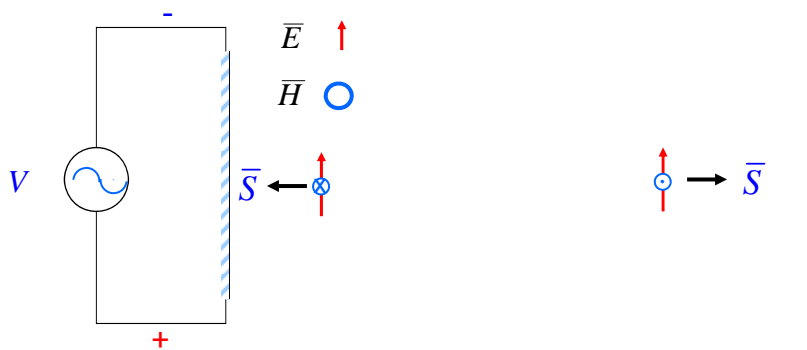
Medium	$\epsilon_r$	$\sigma$	Loss Tangent at Freq. (MHz)		
			0.5	100	1000
Wood	2.1	3.3E-09	5.7E-05	2.8E-07	2.8E-08
Marble	8	1.0E-05	4.5E-02	2.2E-04	2.2E-05
Dry Soil	3.4	1.0E-03	1.1E+01	5.3E-02	5.3E-03
Fresh Water	81	1.0E-02	4.4E+00	2.2E-02	2.2E-03
Steak	40	1.0E+00	9.0E+02	4.5E+00	4.5E-01
Sea Water	81	4.0E+00	1.8E+03	8.9E+00	8.9E-01
Copper	1	5.8E+07	2.1E+12	1.0E+10	1.0E+09

$$\frac{\sigma}{\omega\epsilon} \gg 1 \Rightarrow \text{Good Conductor}$$

$$\frac{\sigma}{\omega\epsilon} \ll 1 \Rightarrow \text{Good Insulator}$$

## Surface Currents and Skin Depth

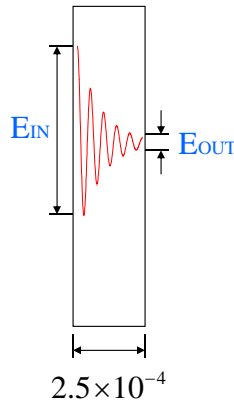
An electric field will create surface currents on a conductor, whether that electric field is created by a voltage source or an incident electromagnetic wave.



The rate at which the fields decay in the conductor, and hence their associated current density will be the same in both cases. Thus, we can use our equation for the field in a lossy medium to find the current density in a wire carrying current,  $I$ .

## Practical Loss Example

If a screen room uses copper sheeting that is 0.25 mm thick, find the lowest frequency that will be attenuated by at least 100 dB as a result of propagating through the copper ( $\sigma = 5.8 \times 10^7$  S/m)



$$\text{We want } 20 \log \left( \frac{E_{OUT}}{E_{IN}} \right) \leq -100$$

$$\Rightarrow 20 \log \left( \frac{E_0 e^{-\alpha 2.5 \times 10^{-4}}}{E_0} \right) \leq -100$$

$$\Rightarrow \alpha 2.5 \times 10^{-4} \log_{10} e = 5 \quad \text{at lowest frequency}$$

$$\Rightarrow \alpha = 46051 \text{ Nepers / m}$$

For a good conductor ( $\sigma / \omega \epsilon \gg 1$ ):

$$\alpha \approx \sqrt{\pi f \mu \sigma} = \sqrt{\frac{\omega \mu \sigma}{2}} = 46051 \Rightarrow f = 9.275 \text{ MHz}$$

General comment: losses due to conduction tend to increase with frequency  $\Rightarrow$  lossy materials tend to act like low-pass filters

## Another Loss Example

Calculate attenuation (dB/m) for underwater radio communications in seawater ( $\epsilon_r = 81$ ,  $\sigma = 4$  S/m)

At FM Band Frequencies (100 MHz)

$$\begin{aligned} \alpha &= \frac{\omega \sqrt{\mu \epsilon}}{\sqrt{2}} \left[ \sqrt{1 + \left( \frac{\sigma}{\omega \epsilon} \right)^2} - 1 \right]^{1/2} \\ &= \frac{(2 \times 10^8 \pi) \sqrt{(4 \pi \times 10^{-7})(81 \times 8.85 \times 10^{-12})}}{\sqrt{2}} \left[ \sqrt{1 + \left( \frac{4}{(2 \times \pi \times 10^8)(81 \times 8.85 \times 10^{-12})} \right)^2} - 1 \right]^{1/2} = 37.4 \text{ Nepers / m} \end{aligned}$$

$$\text{Thus: } |\bar{E}(z)| = E_0 e^{-37.4z}$$

where  $z$  is the distance traveled in meters in the seawater. After traveling one meter, the signal will be  $e^{-37.4} = 5.72 \times 10^{-17}$  times its original value. Putting this in terms of dB:

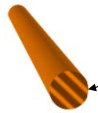
$$dB = 20 \log \left( \frac{|E(z=1)|}{|E(z=0)|} \right) = 20 \log (5.72 \times 10^{-17}) \Rightarrow 324 \text{ dB / m}$$

At 1 kHz, the loss reduces to 1.09 dB/m. In general, the higher the frequency, the greater the loss per unit length.

## Calculating Resistance on a Conductor

The resistance of a wire depends upon its size, frequency, and conductivity:

Conductor	Conductivity, $\sigma$ (S/m)	Resistivity ( $\Omega\text{-cm}$ )
Gold	$3.25 \times 10^6$	$30.8 \times 10^{-6}$
Nickel	$1.45 \times 10^7$	$6.9 \times 10^{-6}$
Aluminum	$3.82 \times 10^7$	$2.62 \times 10^{-6}$
Copper	$5.8 \times 10^7$	$1.72 \times 10^{-6}$
Silver	$6.17 \times 10^7$	$1.62 \times 10^{-6}$



Cross sectional  
area = A

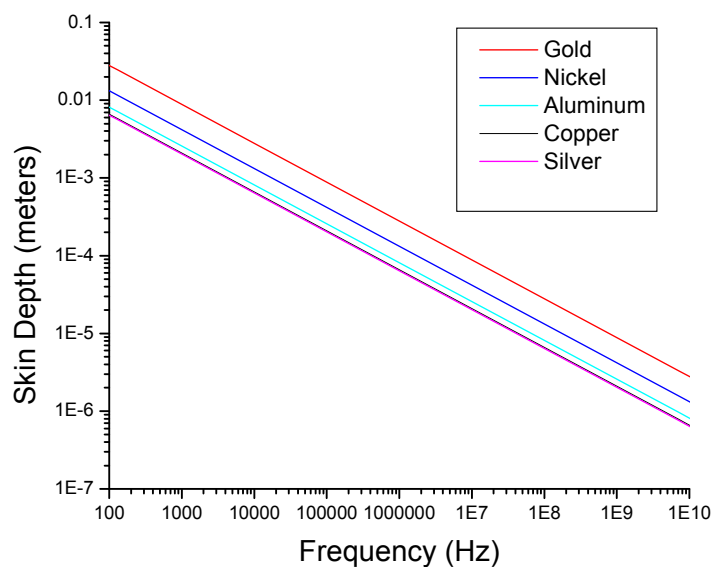
Assuming that current flows uniformly over the entire wire cross section, the conductance per unit length is given by:

$$\text{Conductivity/m} = \sigma (\text{S/m}) A (\text{m}^2) = \sigma A (\text{S-m})$$

$$\Rightarrow \text{Resistivity/m} = 1/G = \frac{1}{\sigma A} \Omega/\text{m}$$

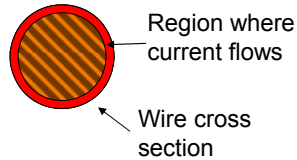
## Conductivity, Frequency & Penetration Depth

As conductivity and/or frequency increase, currents flow closer to the surface of a conductor



## Effect of Penetration Depth on Resistance

Since current does not flow uniformly on most wires, we should account for the current distribution when calculating resistance per unit length.



Example: knowing that the skin depth in copper at 1 GHz is 2.1 microns ( $10^{-6}$  m), estimate the resistance per meter of a 50 micron radius wire.

Assume that we have a uniform current density in the region  $(a - \delta_s) < r < a$  where  $a$  is the wire radius and  $\delta_s$  is the penetration depth.

The area where current flows is then  $\pi a^2 - \pi(a - \delta_s)^2$   
 $= \pi((5 \times 10^{-5})^2 - (4.79 \times 10^{-5})^2) = 6.46 \times 10^{-10} \text{ m}^2$ , which is about 8% of the wire cross sectional area.

$$\text{Resistivity/m} = \frac{1}{\sigma A} = \frac{1}{(5.8 \times 10^7)(6.46 \times 10^{-10})} = 26.7 \text{ } \Omega/\text{m}$$

## Analysis of Penetration Depth on Resistance

In the previous example, we estimated the effect of skin effect by assuming that current traveled uniformly in the penetration depth region. We will check the validity of that assumption below by integrating the current density in the wire.

$$E(r) = E_0 e^{-\alpha(a-r)} \Rightarrow J(r) = E_0 \sigma e^{\alpha(r-a)}$$

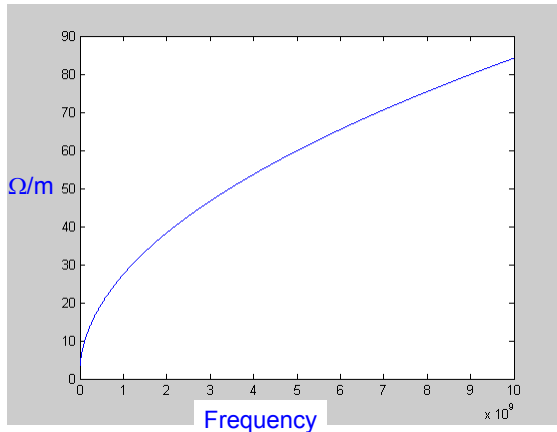
Where  $E_0$  is the applied field (volts/meter) on the surface of the wire. By integrating the current density over the cross section, we can find the total current for our applied voltage per meter:

$$\begin{aligned} I &= \iiint_{\text{wire cross section}} J(s) ds = \sigma E_0 \int_0^{2\pi} \int_0^a e^{-\alpha(a-r)} r dr d\phi = 2\pi \sigma E_0 e^{-\alpha a} \int_0^a e^{\alpha r} r dr \\ &= \frac{2\pi \sigma E_0}{\alpha} \left[ a + \frac{1}{\alpha} (e^{-\alpha a} - 1) \right] = 0.0367 E_0 \text{ Amps} \end{aligned}$$

$$\text{The resistance for this 1 m segment is: } R = \frac{V}{I} = \frac{E_0(1)}{0.0367 E_0} = 27.2 \text{ } \Omega/\text{m}$$

## Using Matlab to Look at Other Frequencies

Resistance per meter for a 50 micron copper wire



### Matlab Code:

```
clear all;
freq = 0:5e6:1e10;
mu=pi*4e-7;
sigma=5.8e7;
alpha=sqrt((pi*mu*sigma).*freq);
a=50e-6;
area = pi*a^2;
```

```
active=(2.*exp(-a.*alpha)./a^2).*((exp(alpha.*a)./alpha.^2).*(alpha.*a)-1)+(1./alpha.^2);
resistivity=1./(area.*active*sigma);
plot(freq,resistivity);
```

## Comment: Special Properties of $e^{-1}$

We use  $e^{-1}$  to describe a number of parameters such as the time constant, standard deviation and skin depth. This provides us with a convenient way with coming up with the exact answer without having to integrate in many cases, such as the previous example. As an example, find the total current flowing on a length  $L$  of a conducting material if the current density is given by:

$$J_0 e^{-\alpha x} \hat{a}_y \text{ Amps/m}^2$$

$$I = \iint \vec{J} \cdot \vec{ds} = \int_0^L \int_0^\infty J_0 e^{-\alpha x} \hat{a}_y \cdot \hat{a}_y dx dz = J_0 L \int_0^\infty e^{-\alpha x} dx$$

$$= \frac{-J_0 L}{\alpha} e^{-\alpha x} \Big|_0^\infty = \frac{J_0 L}{\alpha} = J_0 L \delta$$

Assuming a uniform current distribution within the skin depth and zero current distribution elsewhere provides us with the exact answer.

