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Vector Coordinate Conversion: Cylindrical

There are times when we have a vector field defined in one coordinate system that interacts on a geometry defined in another coordinate system. In this case, we will need to perform a conversion so that both are in the same coordinate system.

For example, say that we are given \( \vec{A} = A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z \) and we would like to represent it as \( \vec{A} = A_r \hat{a}_r + A_\phi \hat{a}_\phi + A_z \hat{a}_z \)

We can find the \( x \)-component \( (A_x) \) using the dot product

\[
A_x = \vec{A} \cdot \hat{a}_x = A_x \hat{a}_x \cdot \hat{a}_x + A_y \hat{a}_y \cdot \hat{a}_x + A_z \hat{a}_z \cdot \hat{a}_x \quad \text{note} \quad \hat{a}_x \cdot \hat{a}_x = 0
\]

\[
\hat{a}_r \cdot \hat{a}_x = \cos \phi \quad \text{and} \quad \hat{a}_\phi \cdot \hat{a}_x = -\sin \phi
\]

Note: if \( \phi = 0^\circ \), then \( \hat{a}_\phi = \hat{a}_y \)
and if \( \phi = 90^\circ \), then \( \hat{a}_\phi = -\hat{a}_y \)

Similarly

\[
A_y = \vec{A} \cdot \hat{a}_y = A_x \hat{a}_x \cdot \hat{a}_y + A_y \hat{a}_y \cdot \hat{a}_y + A_z \hat{a}_z \cdot \hat{a}_y
\]

where \( \hat{a}_r \cdot \hat{a}_y = \sin(\phi) \quad \hat{a}_\phi \cdot \hat{a}_y = \cos(\phi) \)

This conversion can be put in matrix form:

\[
\begin{bmatrix}
A_x \\
A_y \\
A_z
\end{bmatrix}
= \begin{bmatrix}
\cos(\phi) & -\sin(\phi) & 0 \\
\sin(\phi) & \cos(\phi) & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
A_r \\
A_\phi \\
A_z
\end{bmatrix}
\]

Going from rectangular to cylindrical can be found by taking the inverse of the matrix
Spherical Coordinates

Coordinate Conversion

\[ (R, \theta, \phi) \]
\[ x = R \sin \theta \cos \phi \]
\[ y = R \sin \theta \sin \phi \]
\[ z = R \cos \theta \]

Vector coordinate conversion: Spherical-Cartesian

\[
\begin{bmatrix}
A_r \\
A_\theta \\
A_\phi
\end{bmatrix} =
\begin{bmatrix}
\sin \theta \cos \phi & \cos \theta \cos \phi & -\sin \phi \\
\sin \theta \sin \phi & \cos \theta \sin \phi & \cos \phi \\
\cos \theta & -\sin \theta & 0
\end{bmatrix}
\begin{bmatrix}
A_x \\
A_y \\
A_z
\end{bmatrix}
\]

Note that the directions of \( \hat{a}_r, \hat{a}_\theta \) and \( \hat{a}_\phi \) change with \( \theta \) and \( \phi \)

Example Using Coordinate Conversion

Evaluate \( \int_A^B \vec{F} \cdot d\vec{\ell} \) for the contour shown, where:

\[ \vec{F} = xy \hat{a}_x - 2x \hat{a}_y \]
\[ d\vec{\ell} = r d\phi \hat{a}_\phi = 3 \hat{a}_\phi \]

Math will be easier if both were in the same coordinate system- put \( \vec{F} \) in polar:

\[
\begin{bmatrix}
F_r \\
F_\theta \\
F_\phi
\end{bmatrix} =
\begin{bmatrix}
\cos(\phi) & \sin(\phi) & 0 \\
-\sin(\phi) & \cos(\phi) & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
-2x
\end{bmatrix}
\]

\[ \vec{F} = (xy \cos \phi - 2x \sin \phi) \hat{a}_r - (x \sin \phi + 2 \cos \phi) \hat{a}_\phi \]

Substituting for \( x \) and \( y \) : \( x = 3 \cos \phi \) and \( y = 3 \sin \phi \):

\[ \vec{F} = (9 \cos^2 \phi \sin \phi - 6 \cos \phi \sin \phi) \hat{a}_r - (9 \cos \phi \sin^2 \phi + 6 \cos^2 \phi) \hat{a}_\phi \]

\[
\int_C \vec{F} \cdot d\vec{\ell} = - \int_0^{\pi/2} 27 \cos \phi \sin^2 \phi d\phi \cdot \int_0^{\pi/2} \cos^2 \phi d\phi = -9(1 + \frac{\pi}{2})
\]

where we used \( \cos^2 \alpha = \frac{1}{2}(1 + \cos 2\alpha) \) to solve the last integral.
Another Vector Contour Integral Example

Find the work required to travel completely around the contour defined by the intersection of the sphere \( x^2+y^2+z^2 = 10 \) and the plane \( x = -1 \) if \( \vec{F} = 2y^2 \hat{a}_x + y^3 x \hat{a}_y + 4xy \hat{a}_z \) Newtons

The intersection forms the circle \( y^2+z^2 = 9 \) in the \( x = -1 \) plane and \( \vec{F} = 2y^2 \hat{a}_x - y^3 \hat{a}_y - 4y \hat{a}_z \) Newtons

Since the path is circular, it is probably easier to use polar coordinates:

\[
\overline{d\ell} = rd\phi \hat{a}_\phi = 3d\phi \hat{a}_\phi
\]

Note: this is not the standard polar system

Coordinate Conversion:

\[
y = r \cos \phi = 3 \cos \phi \\
z = r \sin \phi = 3 \sin \phi
\]

Another Vector Contour Integral Example (2)

We need only find the \( \hat{a}_\phi \) component of \( \vec{F} \), since \( \overline{d\ell} \) is in only the \( \hat{a}_\phi \) direction

\[
F_\phi = \vec{F} \cdot \hat{a}_\phi = 2y^2 \hat{a}_x \cdot \hat{a}_\phi - y^3 \hat{a}_y \cdot \hat{a}_\phi - 4y \hat{a}_z \cdot \hat{a}_\phi \text{ Newtons}
\]

\[
= y^3 \sin \phi - 4y \cos \phi = 27 \cos^3 \phi \sin \phi - 12 \cos^2 \phi
\]

work = \( \oint \vec{F} \cdot d\overline{\ell} = \int_0^{2\pi} \left( 27 \cos^3 \phi \sin \phi - 12 \cos^2 \phi \right) 3d\phi = \pm 36\pi 
\]

+ or − depending upon which way you integrate around the contour
**Complex Number-Phasor Relationship**

\[ A \cos(\omega t + \phi) \] can be written as \[ A \text{Re}\{e^{j(\omega t + \phi)}\} \]

since Euler tells us that \[ e^{j(\omega t + \phi)} = \cos(\omega t + \phi) + j \sin(\omega t + \phi) \]

By convention, the \( \text{Re}\{ \} \) and the \( e^{j\omega t} \) are assumed, and not written, hence: \[ A \cos(\omega t + \phi) \Rightarrow A e^{j\phi} \] in phasor notation

Representing a phasor as a complex number:

\[ A e^{j\phi} = \{ A \cos \phi, A \sin \phi \} \]

\[
\begin{align*}
\text{Real Part} & \quad \text{Imaginary Part} \\
\text{Re} & \quad \text{Im}
\end{align*}
\]

The sum of a series of sine waves can be determined by summing the complex numbers representing them.

**Complex Number Properties**

\[ |A e^{j\theta}| = \sqrt{\text{Real}^2 + \text{Imag}^2} \]

and phase = \( \phi = \tan^{-1}\left(\frac{\text{Imag}}{\text{Real}}\right) \)

Time derivative of a phasor: if \( B \) is a phasor, then:

\[
\frac{dB}{dt} \left[ e^{j\omega t} \right] = j\omega B \quad \text{and} \quad \frac{d^n B}{dt^n} = (j\omega)^n B
\]

This is often used for the time derivative in Maxwell's equations:

\[ \nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} = -j\omega \mu \vec{H} \]

Note: \( e^{j\omega t} \) and \( e^{j\phi} \) are phase terms, and do not affect the magnitude unless dealing with complex angle (\( \phi \)) or frequency (\( \omega \)).

\[ |e^{j\omega t}| = |e^{j\phi}| = 1 \]
The Del Operator

The Del Operator ($\nabla$) has the properties of a vector, and a differential operator:

$$\nabla = \frac{\partial}{\partial x} \hat{a}_x + \frac{\partial}{\partial y} \hat{a}_y + \frac{\partial}{\partial z} \hat{a}_z$$

in Cartesian coordinates

The del operator is important to us since it provides a number of indications as to how vector and scalar functions vary with position. It shows up in the gradient, curl, divergence, and Laplacian.

Gradient

The gradient of a scalar function of position results in a vector that points in the direction of greatest increase for that function. The magnitude of the gradient indicates how quickly that function changes with position.

If $\Phi(x,y,z)$ is a scalar function of position in Cartesian Coordinates

then $\nabla \Phi(x,y,z) = \frac{\partial \Phi(x,y,z)}{\partial x} \hat{a}_x + \frac{\partial \Phi(x,y,z)}{\partial y} \hat{a}_y + \frac{\partial \Phi(x,y,z)}{\partial z} \hat{a}_z$

Example

If the temperature in a room is $T(x,y,z) = (65 + 0.1x - 0.25y + 0.5z)^{\circ}C$

in what direction is the temperature increasing most rapidly, and what is the rate of change with distance in that direction?

Solution: $\nabla T(x,y,z) = (0.1\hat{a}_x - 0.25\hat{a}_y + 0.5\hat{a}_z)^{\circ}C / m$

Rate of change $= |\nabla T(x,y,z)| = 0.57^{\circ}C / m$
Gradient (2)

The normal to any surface defined by \( f(x,y,z) = \text{constant} \) is given by the gradient of \( f(x,y,z) \)

Example: find the unit normal to the sphere \( x^2 + y^2 + z^2 = 9 \) in Cartesian coordinates. In this case \( f(x, y, z) = x^2 + y^2 + z^2 \), so \( \nabla f(x, y, z) = 2x\hat{a}_x + 2y\hat{a}_y + 2z\hat{a}_z \), and the unit normal is:

\[
\frac{\nabla f(x, y, z)}{|\nabla f(x, y, z)|} = \frac{2x\hat{a}_x + 2y\hat{a}_y + 2z\hat{a}_z}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \frac{x\hat{a}_x + y\hat{a}_y + z\hat{a}_z}{\sqrt{x^2 + y^2 + z^2}}
\]

since \( x^2 + y^2 + z^2 = 9 \) on the spherical surface

Curl

If \( \vec{F} \) represents fluid velocity, then \( \nabla \times \vec{F} \) results in a vector defining the paddle-wheel axis orientation that will result in maximum rotation. Obeys the right hand rule: \( \nabla \times \vec{F} \) points into the screen.

In Cartesian coordinates

\[
\nabla \times \vec{F} = \begin{vmatrix}
\hat{a}_x & \hat{a}_y & \hat{a}_z \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
F_x & F_y & F_z
\end{vmatrix}
\]
How the Curl Works

Consider only the \( z \)-component of curl, which corresponds to the paddlewheel oriented as shown. That wheel will turn only if the \( x \) component of the vector changes with \( y \), or the \( y \) component changes with \( x \). The \( z \) component of the curl is given by:

\[
\begin{vmatrix}
\hat{a}_x & \hat{a}_y & \hat{a}_z \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
F_x & F_y & F_z
\end{vmatrix} = \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}
\]

Curl Examples

\[ \nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \]

\[ H(t) \]

Definition of the curl

\[
\nabla \times \vec{A} \equiv \lim_{\Delta s \to 0} \left[ \hat{a}_n \oint_{\Delta s} \vec{A} \cdot d\ell \right]_{\text{max}}
\]

where \( \Delta s \) is the surface enclosed by the closed contour and \( \hat{a}_n \) is the unit normal to that surface. The curl is a measure of rotation per unit area.
Divergence

Divergence is a measure of compression or decompression of a field. It indicates the field leaving a volume element.

\[ \nabla \cdot \mathbf{F} \text{ is a scalar } = \left( \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right) \mathbf{\hat{a}_x} \mathbf{\hat{a}_y} \mathbf{\hat{a}_z} \]

\[ = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \text{ in Cartesian coordinates} \]

Because the divergence is a differential operator, it tells us something about the vector field at a particular point in space.

If the divergence of a vector field is positive throughout a region, it indicates that more field is leaving that region than entering.

The divergence can only be performed on a vector.

How Divergence Works

Consider the net volume leaving the differential volume element shown if gas velocity \( \mathbf{V} = V_x \mathbf{\hat{a}_x} + V_y \mathbf{\hat{a}_y} + V_z \mathbf{\hat{a}_z} \) m/sec.

Consider the volume of gas leaving each face

Left Face:

\[ -V_y \Delta y \Delta x \Delta z \text{ m}^3/\text{sec} \]

Right Face:

\[ \left[ V_y + \frac{\partial V_y}{\partial y} \right] \Delta x \Delta z \text{ m}^3/\text{sec} \]

Dr. Kent Chamberlin
How Divergence Works (2)

Rear Face: \(-V_y \Delta y \Delta z \text{m}^3/\text{sec}^3\)

Front Face: \[ V_x + \frac{\partial V_x}{\partial x} \Delta x \Delta y \Delta z \text{m}^3/\text{sec}^3 \]

Bottom Face: \(-V_x \Delta x \Delta y \text{m}^3/\text{sec}^3\)

Top Face: \[ V_z + \frac{\partial V_z}{\partial z} \Delta x \Delta y \text{m}^3/\text{sec}^3 \]

The net loss per unit volume is (summing the above)

\[
\frac{\partial V_x}{\partial x} \Delta x \Delta y \Delta z + \frac{\partial V_y}{\partial y} \Delta x \Delta y \Delta z + \frac{\partial V_z}{\partial z} \Delta x \Delta y \Delta z
\]

\[\Delta x \Delta y \Delta z\]

\[= \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = \nabla \cdot \vec{V}\]

Divergence Examples

If gas velocity \(\vec{V} = y^2 \hat{a}_y\) is the region in compression or decompression?

Since \(\nabla \cdot \vec{V} = 2y\) decompression increases with \(|y|\)

If gas velocity \(\vec{V} = x \hat{a}_x + y \hat{a}_y + z \hat{a}_z\) is the region in compression or decompression?

Since \(\nabla \cdot \vec{V} = 3\), decompression is the same everywhere.
Using Divergence in an Equation

If $\vec{V}$ is air velocity, $\Phi$ is air pressure, and $k$ is a constant, then

$$\nabla \cdot \vec{V} = -k \frac{\partial \Phi}{\partial t} \quad \text{As air leaves a region, its pressure decreases}$$

If $\vec{H}$ is heat flow, $T$ is temperature, and $k$ is a constant, then

$$\nabla \cdot \vec{H} = -k \frac{\partial T}{\partial t} \quad \text{As heat leaves a region, it cools off}$$

If $\vec{J}$ is current density, and $\varphi_{ev}$ is charge density, then

$$\nabla \cdot \vec{J} = -\frac{\partial \varphi_{ev}}{\partial t} \quad \text{As current leaves a region, its charge density decreases}$$

If $\vec{E}$ is electric field, and $\varphi_{ev}$ is charge density, then

$$\nabla \cdot \varepsilon \vec{E} = \varphi_{ev} \quad \text{If a region contains an electric charge density, an electric field will emanate from that region}$$

Laplacian

The Laplacian of a scalar function at a particular point indicates how the value of the function at that point compares with its value in surrounding regions.

If $T(x, y, z)$ is temperature in the region, then the Laplacian of $T = \nabla^2 T$ will be positive at $P$, since the surrounding temperatures are higher.

$$\nabla^2 T = \nabla \cdot \nabla T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \quad \text{in Cartesian coordinates}$$
Laplacian: How it works

Consider a 1-dimensional example, where temperature varies only as a function of one direction \( x \Rightarrow T(x) \) with no variation in \( y \) or \( z \). In this case:

\[
\nabla^2 T = \frac{\partial^2 T}{\partial x^2} 
\]

Laplacian in an Equation

The Heat Transfer Equation:

\[
\nabla^2 T = k \frac{\partial T}{\partial t}
\]

where \( T \) is temperature and \( k \) is thermal resistivity

rearranging: \( \frac{\partial T}{\partial t} = \frac{\nabla^2 T}{k} \)

In words: If the surrounding region is warmer than something, that thing will get warmer over time
The Diffusion Equation

\[ \nabla^2 \rho = k \frac{\partial \rho}{\partial t} \] where \( \rho \) is particulate density and \( k \) is the diffusion coefficient

In words: If the particulate density in surrounding region is greater than it is where you are, then it will increase where you are over time.

The Diffusion Equation In Action

Once upon a time, there were three factories operating near a river, and at least one of them was polluting

Factory A  Factory B  Factory C

Moral of story: don’t underestimate the power of differential equations