Reflection and Transmission of a Plane Wave Incident on a Planar Surface

- Since plane waves in space and waves on T-lines are TEM, the reflection and transmission coefficients we have derived apply to both.
- To be consistent with the book, we will use $k$ as the wavenumber rather than $\gamma$.
- We will now look at plane wave reflection from planar surfaces, where the wave may be arriving at angles other than normal.

**Defining Direction of Propagation**

In the notation we have used so far in this class, a plane wave with a $y$-directed $E$-field propagating in the $z$ direction is given by:

$$E_y(z) = E_0 e^{-jkz}$$

A plane wave traveling in any direction is given by:

$$\vec{E}(x, y, z) = \vec{E}_0 e^{-j(k_x x + k_y y + k_z z)}$$

where the direction of propagation is defined by $k_x$, $k_y$, and $k_z$. A condition that must be satisfied is:

$$k_x^2 + k_y^2 + k_z^2 = k^2$$

where $k = \omega \sqrt{\mu \varepsilon}$ as before. If propagation is in the $z$ direction $k_x = k_y = 0$ and $k_z = k$. 

![Diagram showing direction of propagation](image-url)
Defining Direction of Propagation (2)

A common way to represent plane waves traveling at oblique angles (i.e., not aligned with one of the coordinate axes) is:

\[ \vec{E}(\vec{R}) = E_0 e^{-j\vec{k} \cdot \vec{R}} \]

where \( \vec{k} = k_x \hat{a}_x + k_y \hat{a}_y + k_z \hat{a}_z \) and \( \vec{R} = x \hat{a}_x + y \hat{a}_y + z \hat{a}_z \)

Note: \( \vec{k} \cdot \vec{R} = k_x x + k_y y + k_z z \) consistent with the form above

\[ \vec{k} \] points in the direction of propagation

\[ \vec{R} \] is a vector from the origin to \((x, y, z)\)

\[ \vec{E}(\vec{R}) \Rightarrow \vec{E}(x, y, z) \]

Example 1

Find the vector wavenumber for a wave whose magnetic field component is given below:

\[ \vec{H} = (-\hat{a}_x + 3 \hat{a}_y + 2 \hat{a}_z) e^{-j\pi(2x + 2y - 2z)} \]

The phase is defined by the scalar function:

\[ \Phi(x, y, z) = -\pi(2x + 2y - 2z) \]

The planes defined by this function being equal to a constant are the planes of equal phase, as shown in green on the previous slide.

The direction in which this function decreases most rapidly, determined by the gradient, will be in the direction of propagation. The magnitude of that gradient will give the phase change per unit length in the direction of propagation:

\[ \vec{k} = -\nabla \Phi(x, y, z) = \pi \left( 2 \hat{a}_x + 2 \hat{a}_y - 2 \hat{a}_z \right) \]
Example 2

Find the vector wavenumber for a 400 MHz plane wave in a pure dielectric ($\varepsilon_r = 9$) that is propagating in the direction of the unit vector below:

$$\hat{a} = \frac{2\hat{a}_x + \hat{a}_y - 2\hat{a}_z}{3}$$

Solution:

$$k = \frac{2\pi}{\lambda} = \frac{2\pi f}{v_p} = \frac{2\pi \left(4 \times 10^8\right)}{\left(3 \times 10^8\right)} = \frac{8\pi}{3} \text{ radians} / m$$

Because the direction of $k$ is the same as the direction of propagation:

$$\bar{k} = k\hat{a} = \frac{8\pi}{3} \left(2\hat{a}_x + \hat{a}_y - 2\hat{a}_z\right)$$

Determining Phase Change in a Given Direction

We can also use the vector $k$ to determine how quickly the phase of a plane wave is changing in a specified direction by dotting the vector $k$ with the unit vector defining that direction.

For example, in our previous example, how quickly is phase changing in the direction of propagation (which we already know)?

Phase change = $k \cdot \hat{a}_{\text{given}} = \frac{8\pi}{3} \left(2\hat{a}_x + \hat{a}_y - 2\hat{a}_z\right) \cdot \frac{1}{3} \left(2\hat{a}_x + \hat{a}_y - 2\hat{a}_z\right) = 8\pi$

The rate of phase change will be greatest in the direction of propagation, and it will be zero perpendicular to the direction of propagation.
Rate of Phase Change on a Boundary

If a plane wave is incident on a boundary as shown, how quickly will the phase of that wave change along the boundary? We will need this to find the reflected and transmitted fields.

Phase change of the incident field on the boundary = $k_1 \sin \theta_i$ radians/m

Because the boundary is illuminated by a plane wave, the total field on the boundary must be constant (i.e., uniform). This will happen only if the phase of the reflected field changes at the same rate as the incident field.

This implies that: $k_1 \sin \theta_i = k_1 \sin \theta_r \Rightarrow \theta_i = \theta_r$

The angle of incidence is equal to the angle of reflection

Snell’s Law of Refraction

Similarly, the transmitted field phase must also change at the same rate as the incident and reflected fields:

$$k_1 \sin \theta_i = k_1 \sin \theta_r = k_2 \sin \theta_t$$

$\theta_i$, $\theta_r$, and $\theta_t$ are the angles of incidence, reflection, and transmission. Rearranging, we get:

$$\sin \theta_i = \frac{k_1}{k_2} \sin \theta_t$$

This is one form of Snell’s Law of refraction. Knowing the angle of incidence, we can find the angle of transmission.
Snell’s Law Example

If region 1 is free space, and region 2 is a dielectric with \( \varepsilon_r = 9 \), find the angle of transmission into region 2 for a plane wave incident at an angle of 50°.

**Solution**

\[
\sin \theta_i = \frac{k_1}{k_2} \sin \theta_i = \frac{\omega \sqrt{\mu_r \varepsilon_i}}{\omega \sqrt{\mu_r \varepsilon_2}} \sin \theta_i = \frac{\omega \sqrt{\mu_0 \varepsilon_0}}{\omega \sqrt{\mu_0 (9 \varepsilon_0)}} \sin 50°
\]

\[
\sin \theta_i = \frac{1}{3} \sin 50° = 0.2553 \Rightarrow \theta_i = \sin^{-1} (0.2553) = 14.8°
\]

If permittivity changes with frequency, different frequencies will be transmitted at different angles, which is why prisms are able to separate the different colors (frequencies) of light.

**Reflection and Transmission (oblique incidence)**

To apply boundary conditions, we need to identify 2 cases, one where \( E \) is parallel to the boundary, and the other where \( H \) is parallel to the boundary.

**Case 1:** \( E \) parallel to the boundary - called horizontal or perpendicular polarization, because \( E \) is normal to the plane formed by the normal to the boundary and the direction of propagation. Applying boundary conditions:

\[
E_{tan_1} = E_{tan_2} \Rightarrow E'_y + E'_y = E'_y
\]

\[
H_{tan_1} = H_{tan_2} \Rightarrow H'_x - H'_x = H'_x \Rightarrow H' \cos \theta_i - H' \cos \theta_i = H' \cos \theta_i
\]

noting that \( \theta_i = \theta_r \), \( \eta_1 = \frac{E_i}{H_i} = \frac{E_r}{H_r} \) and \( \eta_2 = \frac{E_i}{H_i} \) we can put our equation for \( H_{tan} \) in terms of \( E \):

\[
\frac{E_i}{\eta_i} \cos \theta_i - \frac{E_r}{\eta_i} \cos \theta_i = \frac{E_i}{\eta_2} \cos \theta_i \Rightarrow E_i - E_r = E_i \frac{\eta_i \cos \theta_i}{\eta_2 \cos \theta_i}
\]
Knowing that $E_i + E_r = E_t$, we can substitute for $E_t$ in the above equation, which will give us the relationship between $E_i$ and $E_r$. Defining the reflection coefficient, $\Gamma$, as $E_r/E_i$:

$$\Gamma = \frac{\eta_2 \cos \theta_i - \eta_1 \cos \theta_i}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_i}$$

we can find $\theta_i$ from $\theta_i$ using Snell's Law.

Note that this will give us the same reflection coefficient that we obtained when solving for normal incidence, where $\theta_i = \theta_t = 0^\circ$.

**Parallel, or Vertical Polarization**

**Case II**: $H$ parallel to the boundary—called vertical or parallel polarization, because $E$ is parallel to the plane formed by the normal to the boundary and the direction of propagation. Using the same type of analysis as for horizontal polarization, we get the reflection coefficient:

$$\Gamma = \frac{\eta_2 \cos \theta_i - \eta_1 \cos \theta_i}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_i}$$

Definition: **Brewster Angle**—with vertical polarization, there are cases where $\Gamma$ goes to zero $\Rightarrow$ no energy is reflected and all energy is transmitted into the second medium.
Reflection Coefficients for Average Ground

\[ \varepsilon_r = 15 \quad \text{and} \quad \sigma = 8 \times 10^{-3} \ \text{S/m} \]

Reflection Coefficients for Seawater

\[ \varepsilon_r = 81 \quad \text{and} \quad \sigma = 4 \ \text{S/m} \]
Image With & Without Polarized Lens
Finding the Brewster Angle

**Vertical Polarization** Since the Brewster angle occurs at the angle where the reflection coefficient goes to zero, we need to solve for:

\[
\Gamma = \frac{\eta_2 \cos \theta_i - \eta_1 \cos \theta_i}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_i} = 0 \Rightarrow \eta_2 \cos \theta_i = \eta_1 \cos \theta_i
\]

To simplify calculations, consider the lossless case and assume \( \mu = \mu_0 \) in both media:

\[
\sigma = 0 \Rightarrow \eta = \sqrt{\frac{\mu_0}{\varepsilon}} \quad \text{Substituting this into the above}
\]

\[
\sqrt{\frac{\mu_0}{\varepsilon_2}} \cos \theta_i = \sqrt{\frac{\mu_0}{\varepsilon_1}} \cos \theta_i \Rightarrow \cos \theta_i = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}} \cos \theta_i
\]

Squaring both sides

\[
\cos^2 \theta_i = \frac{\varepsilon_1}{\varepsilon_2} \left(1 - \sin^2 \theta_i \right) \quad \text{using } \sin^2 \theta_i + \cos^2 \theta_i = 1 \text{ on the right}
\]

Finding the Brewster Angle (2)

Use Snell's Law to put \( \sin \theta_i \) in terms of \( \sin \theta_i \)

\[
\cos^2 \theta_i = \frac{\varepsilon_1}{\varepsilon_2} \left(1 - \frac{\varepsilon_1}{\varepsilon_2} \sin^2 \theta_i \right)
\]

Since \( \cos^2 \theta_i + \sin^2 \theta_i = 1 \), we can use it in place of the 1 above:

\[
\cos^2 \theta_i = \frac{\varepsilon_1}{\varepsilon_2} \left(\cos^2 \theta_i + \sin^2 \theta_i - \frac{\varepsilon_1}{\varepsilon_2} \sin^2 \theta_i \right)
\]

Rearranging:

\[
\sin^2 \theta_i \left(\frac{\varepsilon_1}{\varepsilon_2} - 1\right) = \cos^2 \theta_i \left(1 - \frac{\varepsilon_2}{\varepsilon_1}\right) \Rightarrow \frac{\sin^2 \theta_i}{\cos^2 \theta_i} = \tan^2 \theta_i = \frac{\varepsilon_2}{\varepsilon_1}
\]

At the Brewster Angle \( \theta_i = \theta_B \)

\[
\theta_B = \tan^{-1} \left(\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\right)
\]
Brewster Angle Application

Remote detection of land mines - by illuminating the ground at the Brewster angle, the signal incident on the mine will be maximized.

A broad-band signal is used as the source so that the spectrum of the backscattered field can be used to identify the “signature” of a mine.

**Example:** Assuming average ground ($\varepsilon_r = 1.5$) and ignoring conductivity, what is the Brewster angle for the above application?

$$\theta_B = \tan^{-1}\left(\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\right) = \tan^{-1}\left(\sqrt{1.5}\right) = 75.5^\circ$$

Finding the Brewster Angle (2)

**Horizontal Polarization** Again, we solve for the angle where the reflection coefficient goes to zero:

$$\Gamma = \frac{\eta_2 \cos \theta_i - \eta_1 \cos \theta_i}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_i} = 0 \Rightarrow \eta_2 \cos \theta_i = \eta_1 \cos \theta_i$$

And again consider the lossless case:

$$\sigma = 0 \Rightarrow \eta = \sqrt{\frac{\mu}{\varepsilon}}$$

Substituting this into the above

$$\sqrt{\frac{\mu_2}{\varepsilon_2}} \cos \theta_i = \sqrt{\frac{\mu_1}{\varepsilon_1}} \cos \theta_i \Rightarrow \cos \theta_i = \sqrt{\frac{\mu_2}{\mu_1}} \frac{\varepsilon_1}{\varepsilon_2} \cos \theta_i$$

Squaring both sides

$$(1 - \sin^2 \theta_i) = \frac{\mu_1}{\mu_2} \frac{\varepsilon_1}{\varepsilon_2} (1 - \sin^2 \theta_i)$$ and using Snell’s law to represent $\sin \theta_i$

$$(1 - \sin^2 \theta_i) = \frac{\mu_1}{\mu_2} \frac{\varepsilon_1}{\varepsilon_2} \left(1 - \frac{\mu_1}{\mu_2} \frac{\varepsilon_1}{\varepsilon_2} \sin^2 \theta_i \right) \Rightarrow \sin \theta_i = \sqrt{\frac{\varepsilon_2}{\varepsilon_1} - \frac{\mu_1}{\mu_2}}$$

The condition that must be met to keep $\sin \theta_i \leq 1$ is: $\frac{\varepsilon_2}{\varepsilon_1} \leq \frac{\mu_1}{\mu_2}$

When traveling from a less dense to a more dense medium, and if the permeabilities are the same, there is no Brewster angle with horizontal polarization.
**Notation for Working with 2-D Plane Waves**

A plane wave propagating in 2-dimensional space at an angle $\theta$ can be represented by:

$$E(x, y, \theta) = E_0 e^{-jk(x \cos \theta + y \sin \theta)}$$

![Direction of plane-wave propagation](image)

Note that this form references phase to 0° at the origin.

We will use this form to represent oblique incidence and to solve problems.

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**Example Problem**

Two antennae are illuminated by a plane wave that is incident at an angle $\theta$ as shown. The antennas are positioned such that when $\theta = 0^\circ$, the upper antenna will be one-quarter wavelength closer to the source, and when $\theta = 90^\circ$, the bottom antenna will be on e-half wavelength closer to the source as shown. Find the output of the summer, in dB (referenced to the output when the signal from both antennas are in phase) when $\theta = 60^\circ$.

![Incident Plane Wave](image)
Problem Solution

Impose your own coordinate system, referenced to the lower antenna (i.e., assume phase = 0°)

\[
\text{Upper Antenna } (-\frac{\lambda}{4}, \frac{\lambda}{2})
\]

\[
\text{Lower Antenna } (0, 0)
\]

Plugging in the values for \(x, y\), and \(\theta\), we can get the phase of the signal at the upper antenna:

\[
E(-\frac{\lambda}{4}, \frac{\lambda}{2}, 60°) = E_0 e^{-j \frac{2\pi}{\lambda}(-\frac{\lambda}{4}\cos 60° + \frac{\lambda}{2}\sin 60°)}
\]

\[
= E_0 e^{j(\frac{\pi}{2}\cos 60° - \pi\sin 60°)} = E_0 e^{-j\pi(0.616)} = E_0 (-0.356 - j0.934)
\]

Problem Solution

Thus, the output of the summer will be:

\[(1 + j0) + (-0.356 - j0.934) = (0.644 - j0.934)\]

\[|0.644 - j0.934| = 1.134\]

Converting this to dB, referenced to the maximum output:

\[
20\log_{10}\left(\frac{\text{output}}{\text{max output}}\right) = 20\log_{10}\left(\frac{1.134}{2}\right) = -4.9
\]

Note that the difference in distance travelled by the plane wave is \(x \cos \theta + y \sin \theta\)