Comment

When placing a charged conductor near a large, uncharged conductor, a charge distribution will form on the uncharged conductor. If the separation distance between the two conductors is small compared with the dimensions of the smaller conductor, the charge distributions will nearly mirror each other.

The net charge on the larger conductor will not be changed by the charge on the smaller conductor (it will still be zero). The charge distribution on the larger conductor are formed by Coulombic forces

\[ \vec{F} = \frac{Q_1 Q_2}{4\pi\varepsilon R^2} \hat{a}_{21} \]

Surface Charge and Electric Field

A surface charge on a single conductor will produce an electric field that is normal to the surface. That electric field is given by:

\[ \vec{E} = \hat{\alpha} \frac{\rho_s}{2\varepsilon} \quad (\text{Volt/m}) \]

The electric field will remain constant above the plate, so long as the distance above the plate is small with respect to the size of the plate.
Electric Field Between Two Conductors

Side View of two parallel conductors with equal and opposite charge.

In this case, the fields from the two conductors add between the conductors, and cancel outside the conductors. The assumption is that the separation between the conductors is small with respect to the conductor dimensions. The electric field between the conductors is:

\[ \vec{E} = \frac{\rho \hat{a}_r}{\varepsilon} \] between conductors

\[ \vec{E} = 0 \] elsewhere

Example (Gauss Law, Capacitance, and Shielding)

For the co-axial conductors below, find the electric field as a function of \( r \), and the capacitance per meter.

To find the electric field, we will apply Gauss Law

\[ \oint \vec{E} \cdot d\vec{S} = \frac{Q_{\text{enclosed}}}{\varepsilon} \]

Note: development is nearly identical to finding \( E \) for a line charge

The electric field leaving a closed surface is equal to the charge enclosed divided by the permittivity.

By symmetry, we can assume that \( \vec{E} \) will emanate only in the radial direction \( \Rightarrow \vec{E} = E_r \hat{a}_r \)

To make our calculations easier, pick a Gaussian surface that takes advantage of symmetry- a cylindrical surface in this case (the field will be normal to the surface, and it will be constant on the surface).
Example (Gauss Law, Capacitance, and Shielding) (2)

Because $E$ is constant on $S$ and because $E$ is parallel to $d\vec{S}$

\[
\oint E \cdot d\vec{S} = E_r \oint dS = E_r \times \text{area of cylinder} = E_r (2\pi r L)
\]

If we assume a charge $Q$ is distributed on each 1 meter section of conductor (both inner and outer), we can find the resulting electric field. Note that the charges will be surface charges at $r = a$ and $r = b$

For $a < r < b$, the total charge enclosed is $Q$. Using this in Gauss’ Law, we get:

\[
\oint E \cdot d\vec{S} = \frac{Q_{\text{enclosed}}}{\epsilon} \Rightarrow E_r (2\pi r) = \frac{Q}{\epsilon} \quad \text{or} \quad E_r = \frac{Q}{2\pi \epsilon r}
\]

For $r > b$, both conductors are enclosed by the Gaussian surface, so the total charge enclosed is zero if the line is balanced. Consequently, the electric field is zero for $r > b$.

Example (Gauss Law, Capacitance, and Shielding) (3)

**Shielding Consideration:** because the electric field caused by charges inside a balanced co-axial conductors go to zero outside of the outer conductor, it is a desirable configuration to use from a shielding viewpoint.

**Finding capacitance per meter:** We already know the charge stored ($Q$), and we need to find the voltage corresponding to that charge. Recalling that:

\[
V = \int_C \vec{E} \cdot d\vec{l}, \quad \text{the voltage between the conductors is}
\]

\[
V = \int_a^b E_r \, dr = \frac{Q}{2\pi \epsilon} \int_a^b \frac{dr}{r} = \frac{Q}{2\pi \epsilon} \ln \frac{b}{a} = \frac{Q}{2\pi \epsilon} \ln \frac{b}{a}
\]

From this we can find the capacitance for this one-meter section of co-axial conductors:

\[
\frac{C}{m} = \frac{Q}{V} = \frac{2\pi \epsilon}{\ln \frac{b}{a}} \quad \text{farads/meter}
\]
Comment: Electric Field Shielding at DC and Low Frequencies

Unless the charges on the inner and outer conductors are equal and opposite, the total charge enclosed by a Gaussian surface will not be zero outside the outer conductor, and thus the electric field will not be zero. In fact, placing an uncharged conductor around an electric field source will provide no shielding at low frequencies.

![Cross-section diagram showing charged inner conductor and outer conductor with no net charge]

Cancellation of Electric Fields

For wires carrying equal and opposite charge, the electric field will decrease with distance from the wires, except between the wires. The closer the wires are to each other, the more rapid the cancellation.

To achieve good electric field shielding, keep charge-carrying conductors balanced and close together.
**Inductance**

Capacitance represents the total charge stored for an applied voltage, \(V\).
\[
C = \frac{Q}{V} \text{ (Coul/Volt or Farads)}
\]

The dual of capacitance is inductance, and it represents the magnetic flux (\(\psi\)) stored for a current (\(I\))

**Magnetic Field Variables**

- \(\vec{H}\) is the magnetic field (amperes/meter)
- \(\vec{B}\) is the magnetic flux density (Webers/m^2)
- \(\vec{B} = \mu \vec{H}\) constitutive relationship
- \(\mu\) is the permeability of the material (henrys-meter)
- \(\psi\) is the magnetic flux (Webers)

The magnetic flux passing through a surface is given by:
\[
\psi = \iint_{S} \vec{B} \cdot d\vec{S}
\]

**Magnetic Fields**

Static electric fields are created by charge, and static magnetic fields are created by current

The static magnetic field is related to current by Ampere's Law:
\[
\oint_{C} \vec{H} \cdot d\vec{l} = \iint_{S} \vec{J} \cdot d\vec{S} = \text{the current enclosed by the contour } C
\]
Applying Ampere’s Law

Find the magnetic field at a distance \( r \) from a wire carrying a current \( I \).
Looking at a crosssection of the wire, and assuming that current is flowing into the wire, a magnetic field will rotate clockwise around it. Ampere’s Law can be most easily applied by taking advantage of symmetry, so let us use a circular contour centered on the wire axis. This contour is in the same direction as the magnetic field, so:

\[
\mathbf{H} \cdot d\mathbf{l} = H \phi d\mathbf{l}
\]

By symmetry, \( H\phi \) will be constant over our contour, so:

\[
\oint_{C} \mathbf{H} \cdot d\mathbf{l} = \oint_{C} H \phi d\mathbf{l} = \oint_{C} H \phi \times \text{length of contour} = H \phi (2\pi r)
\]

Using this in Ampere’s Law:

\[
\oint_{C} \mathbf{H} \cdot d\mathbf{l} = H \phi (2\pi r) = I_{\text{enclosed}} \Rightarrow H \phi = \frac{I}{2\pi r} \quad \text{(Amps/m)}
\]

Since \( \mathbf{B} = \mu \mathbf{H} \Rightarrow \mathbf{B} = \frac{\mu I \phi}{2\pi r} \quad \text{(Webers/m}^2\text{)}

Example (Ampere’s Law, Inductance, and Shielding)

Calculate the inductance per meter for the coaxial conductors shown. Assume that the inner conductor carries current \( I \), and the outer conductor carries current \(-I\) (i.e., it is balanced).

First, we will need to find the total flux generated by the current per meter.

**Note:** since charges reside on the conductor surfaces at \( r=a \) and \( r=b \), so will the current. These are called surface currents.

As the radius of our contour of integration becomes greater than \( b \), the total current enclosed becomes zero, and hence the magnetic field is zero for \( r > b \).

**Shielding Consideration:** the magnetic field caused by currents inside a balanced co-axial conductors go to zero outside of the outer conductor, thus shielding against interference.
Comment on Line Balancing

A balanced co-axial line has the same current on both inner and outer conductors, and will shield the signal it is carrying. An imbalance can occur when the “ground” on one side of a co-ax line is at a different potential than the “ground” at the other end. **Example:** ground current (can be caused by a range of sources) caused by lightning.

![Diagram of ground currents](image)

Currents flowing in the resistive ground cause the potential at the two “ground” points to be different, which will result in currents in the outer conductor that will unbalance the line.
Example of Electric Arc

High-Voltage Switches & Arcing
Co-Ax Inductance

Calculating the inductance per meter for the co-axial conductors. First, we will need to find the total flux generated by the current. To find the total magnetic flux generated by the current, we need to integrate all of the magnetic flux density passing through any surface between the conductors ($a < r < b$).

Because $\psi = \iiint_B B \cdot ds$ choose a surface that is $\perp$ to $\mathbf{B}$ such as the one shown.

When integrating, we only need to evaluate for $a < r < b$, since $\mathbf{B} = 0$ outside that range:

$$\psi = \iiint_B B \cdot ds = \iiint_B B_r ds = \int_0^b \int_0^1 B_r dz dr = \frac{\mu_0 I}{2\pi} \int_0^1 \frac{1}{r} dr = \frac{\mu_0 I}{2\pi} \ln\left(\frac{b}{a}\right)$$

From this, we can get inductance per meter:

$$L/m = \frac{\psi}{I} = \frac{\mu_0}{2\pi} \ln\left(\frac{b}{a}\right) \text{ Henrys/meter}$$

Cancellation of Magnetic Fields

For wires carrying equal and opposite current, the magnetic field will decrease with distance except between the wires. The closer the wires are to each other, the more rapid the cancellation.
Low-Frequency Magnetic Field Shielding Using \( \mu \)-Metal (a high-permeability material)

Fulgurite - when lightning strikes sand
Sunspots: moving charges in plasma exposed to magnetic fields

Lorentz Force:
\[ F = q(\vec{E} + \vec{V} \times \vec{B}) \]

Finding the Inductance per Meter for Two Parallel Wires

Wire inductance can “choke” high-frequency signals. The long wires on the circuit shown would prevent the transfer of high frequency signals. To get a quantitative feel for the frequency limit, we can model signals carried by parallel wires of radius \(a\), separated by a distance, \(d\), for our circuit model:
Finding the Inductance per Meter for Two Parallel Wires

To calculate the inductance, we need to make some starting assumptions:

1. There are equal and opposite currents in the wires.
2. The wires are close enough together so that the magnetic field cancels at \( x > (d-a) \) and \( x < a \).

\[ \vec{B}_L \text{ from the left wire is } \frac{\mu_l I_a}{2\pi x} \text{ and } \vec{B}_R \text{ from the right wire is } \frac{\mu_l I_y}{2\pi (d-x)} \]

To get the total magnetic flux, we need to integrate the contribution of both wires between \( x = a \) and \( x = d-a \):

\[ \psi / m = \iint (\vec{B}_R + \vec{B}_L) \cdot d\vec{S} = \frac{\mu I}{2\pi} \int_a^{d-a} \left( \frac{1}{x} + \frac{1}{d-x} \right) dx dz = \frac{\mu I}{\pi} \ln \left( \frac{d-a}{a} \right) \]

From this, we can get inductance per meter:

\[ L / m = \frac{\psi}{I} = \frac{\mu}{\pi} \ln \left( \frac{d-a}{a} \right) \text{ Henrys/meter} \]

Example of Parallel-Wire Transmission Line:

300Ω twin-lead
Example

Calculate the effect of wire inductance on our circuit if the signal is carried by 2, 15 cm wires of radius 0.5mm. Assume a separation distance of 2 cm.

Inductance per meter \( L = \frac{\mu}{\pi} \ln \left( \frac{d-a}{a} \right) = \left( \frac{4\pi \times 10^{-7}}{\pi} \right) \ln \left( \frac{0.02 - 0.0005}{0.0005} \right) = 1.46 \ \mu \text{H/m} \)

Thus, our 15 cm wire pair will have an inductance of 0.22 \( \mu \text{H} \), and our circuit can be modeled as:

\[
V_{\text{received}} = \frac{V_{\text{sent}} R_{\text{input}}}{R_{\text{input}} + j\omega L} \quad \Rightarrow \quad V_{\text{received}} = \frac{V_{\text{sent}} R_{\text{input}}}{R_{\text{input}} + j\omega L} = \frac{1}{1 + j(6.9 \times 10^{-13})f}
\]

This inductance by itself will not cause distortion at frequencies of interest (because of the high input impedance), but coupled with stray capacitance it will add to the low-pass filtering effect.

Example: Including Stray Capacitance and Inductance
Finding the Capacitance per Meter for Two Parallel Wires

From our co-axial conductor problem, we calculated the electric field from a cylindrical conductor of radius \( a \):

\[
E_s = \frac{\rho_s}{2\pi\varepsilon r} = \frac{\rho_s (2\pi a)}{2\pi\varepsilon r} = \frac{Q(2\pi a)}{(2\pi a) L 2\pi\varepsilon r} \quad r > a
\]

Assuming a charge \(+Q\) on a one-meter section of the left conductor, and \(-Q\) on the right conductor, we can find \( E_x \) between the wires:

\[
E_x \text{ from the left wire is } \frac{Q}{2\pi L \varepsilon x} \quad \text{and } E_x \text{ from the right wire is } \frac{Q}{2\pi\varepsilon L(d-x)}
\]

We get the voltage between the wires by integrating the total electric field between them:

\[
V = \int_c^d E \cdot d\vec{l} = \frac{Q}{2\pi L} \int_a^d \left[ \frac{1}{x} + \frac{1}{d-x} \right] dx = \frac{Q}{\pi\varepsilon} \ln \left[ \frac{d-a}{a} \right]
\]

Finding the Capacitance per Meter for Two Parallel Wires (2)

Using this, we can find the capacitance:

\[
C = \frac{Q}{V} = \frac{\pi\varepsilon}{\ln \left[ \frac{d-a}{a} \right]} \quad (\text{Farads/meter}) \quad a << d \quad \text{parallel wires}
\]

Note: the above calculations were made assuming \( d >> a \). In this case, the presence of the other wire will not change the charge distribution on the wires. As the wires become close, or the radii become large, the capacitance per meter becomes:

\[
C = \frac{Q}{V} = \frac{\pi\varepsilon}{\cosh^{-1} \left[ \frac{d}{a} \right]} \quad (\text{Farads/meter}) \quad a < d \quad \text{parallel wires}
\]

Coulombic forces change the charge distribution on the conductors.
Surface Current and Magnetic Field

First consider the magnetic field produced by current on a wire:

\[ I \]
\[ H \]

Top View

Now, create a surface by placing wires side by side:

The normal components of magnetic field will cancel, except near the sides of the surface. Thus, except near the sides, there will only be a tangential magnetic field caused by this surface current.

Surface Current and Magnetic Field

If a conducting surface carries a current, it will travel as a surface current, \( J_S \) (Amps/m).

\[ J_S \]
\[ w \]

If the current distribution is uniform (a reasonable assumption here), \( J_S = I/w \) (Amps/m).

This surface current will create a magnetic field that is parallel to the conductor surface. If the plate dimensions are large with respect to the distance from the plate, that magnetic field will remain constant with distance.
Surface Current and Magnetic Field (2)

Thus, the magnetic field created by the surface currents of two parallel conductors carrying opposite and equal currents will be constant between the plates, and will be parallel to the surface.

\[ J_s \]

\[ \mathbf{H} \]

Side view of parallel, current-carrying conductors

To find the magnitude of that field, apply Ampere's Law (look at an end view of one of the conductors):

For our contour, the only place that the field is non-zero is above the conductor, so:

\[ \oint \mathbf{H} \cdot d\mathbf{l} = Hw \]

Traces on PCB Guide High-Frequency Signals
Typical Multilayer PCB Layout

Finding the Inductance per Meter for Two Parallel Plates

\[
L/m = \frac{\psi}{I} \quad \text{Henrys/meter}
\]

\[
\psi / m = \text{flux generated} = \iiint B \cdot d\vec{S} = \mu \iiint H \cdot d\vec{S}
\]

\[
= \mu \iiint \frac{I}{w} dS = \mu \left( \frac{I}{w} \right) a(1)
\]

\[
L/m = \frac{\psi}{I} = \frac{\mu a}{w} \quad \text{Henrys/meter}
\]
Odds & Ends

When estimating the capacitance of twin-lead or twisted-pair lines, use the dielectric permittivity of the insulator between the two wires, since most of the electric field will be confined to the region between the conductors.

Twisted-Pair Transmission Line

The twists do not appreciably affect capacitance or inductance, but they do help to keep the wires together.

They also help to keep the line balanced when it is near a conductor

Because the lines are twisted, the capacitance between each of the lines and the conductor will be the same, thus maintaining balance